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Deformation Quantization of Relativistic Particles in Electromagnetic Fields [1]

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The Weyl-Wigner-Moyal formalism for Dirac second class constrained systems has been proposed recently as the deformation quantization of Dirac bracket. In this paper, after a brief review of this formalism, it is applied to the case of the relativistic free particle. Within this context, the Stratonovich-Weyl quantizer, Weyl correspondence, Moyal $\star$-product and Wigner function in the constrained phase space are obtained. The recent Hamiltonian treatment for constrained systems, whose constraints depend explicitly on time, are used to perform the deformation quantization of the relativistic free charged particle in an arbitrary electromagnetic background. Finally, the system consisting of a charged particle interacting with a dynamical Maxwell field is quantized in this context.

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1. Introduction

At the present time deformation quantization constitutes an alternative method to quantize classical systems. It started with the Weyl correspondence in quantum mechanics between classical observables as the algebra of smooth functions $\mathcal{A}_\Gamma$ on the phase space Γ and linear operators acting over certain Hilbert space $\mathcal{H}$ [1,2,3,4]. This is a one-one correspondence where the (noncommutative) operator product is mapped to an associative and non-commutative product, called the Moyal star product $\star$. This product is defined on the formal series-valued ring of functions $\mathcal{A}_\Gamma[[\hbar]]$ on Γ. This correspondence is today known as the Weyl-Wigner-Moyal (WWM) correspondence or WWM formalism.

The theory of deformation quantization was formulated in more rigorous mathematical setting in the context of deformation theory in Refs. [5,6]. In particular, in Ref. [6] there were described some examples of quantum systems in terms of deformations of the symplectic structure. For instance, the spectrum of the hydrogen atom was computed explicitly. Thus the deformation quantization proved to be equivalent to another quantization methods as the canonical quantization or the Feynman path integrals. This equivalence seems to point out the fact that *deformation is quantization*. Deformation quantization from the physical point of view has been be surveyed recently in Refs. [7].

Later there was found that star product in fact does exist for any symplectic manifold [8]. Moreover, there was an explicit construction, trough the uses of symplectic differential geometry, due Fedosov [9]. More recently, Kontsevich proved that the star product, in general, does exist for any Poisson manifold [10].

The most part of the work on deformation quantization has been carried over for the years to dynamical systems with a finite number of degrees of freedom [11,12,13]. One of the recent applications is, for instance, to the damped oscillator [14]. The other is the extension of deformation quantization to classical systems with a finite number of fermionic degrees of freedom (Grassmann variables) [15,16]. More recently the Weyl correspondence has been playing an important role in the context of open string theory in the presence of a B-field [17].

However, the method of deformation quantization also has been applied to describe quantum fields and strings. To be more precise, in Ref. [18], the problem of the UV

divergences in the vacuum energy was discussed. Further developments can be found in Refs. [19,20,21]. Furthermore, the quantization of bosonic strings was discussed later in Ref. [22]. The extension of deformation quantization to infinite number of fermionic degrees of freedom was studied recently in Ref. [23].

Also it has been applied to gravitational systems and systems with constraints [24]. In particular in the first reference, dynamical systems with first class constraints were studied. In the present paper we continue with this philosophy and we will quantize the relativistic particle for the free and interacting cases. In order to do that we will use the recent results [25,26,27], concerning the WWM formalism for a second class constrained system. For related results on this subject, see Refs. [28,29,30]. Within this prescription of quantization for second class constraints, once we gauge fixing, we can re-express all the formalism in the reduced phase space in terms of the physical variables only. We shall observe that the description can be obtained from the low energy limit $\alpha' = \ell_S^2 \to 0$, of the deformation quantization of the bosonic string worked out at [22].

In general terms, the description of classical constrained systems (for classical reviews, see [31,32,33,34,35]) with constraints depending explicitly on time represent a challenge, even at the classical level. A proposal which changes radically the Hamiltonian form of the theory was introduced in Refs. [36,37]. Recently, a geometrical proposal which preserved the Hamiltonian evolution equation presented on the constrained phase space, also to the physical (reduced) phase space, was done in Ref. [38]. This treatment allowed to quantize the second class constrained system of the relativistic charged particle moving in an arbitrary electromagnetic background [39]. Precisely this procedure is what allows to find a Hamiltonian formulation on the physical phase space in order to prove the consistency of the deformation quantization for second class constrained systems proposed in [25,26,27]. In particular, we use this formalism to quantize by deforming the phase space of a free relativistic particle, in an arbitrary electromagnetic background and interacting with a dynamical electromagnetic field.

Our paper is organized as follows. In Sec. 2 we overview the covariant description of a relativistic free particle. Sec. 3 is devoted to give in general the WWM formalism on the constrained phase space. Thus the Stratonovich-Weyl (SW) quantizer, which is the main

object to determine the Weyl correspondence, is found. The Moyal $\star$-product, Wigner function, some of its properties and correlation functions are also defined. In Sec. 4, we show that the formalism to deal time-dependent second class constraints given at [38,39] are precisely what is needed to perform appropriately the deformation quantization in the constrained phase space from Refs. [25,26,27]. All ingredients of the WWM are computed in the light-cone gauge and it is shown that they coincides with the low-energy effective theory with $\alpha' \to 0$ from the bosonic string [22]. Some properties of the Wigner function concerning its correspondence with a pure state is discussed in an appendix. Also in Sec. 5 the deformation quantization of the particle in an arbitrary electromagnetic background is also described within this formalism. In Sec. 6, we discuss the deformation quantization of the relativistic charged particle interacting with a dynamical electromagnetic field. This is done in the Lorentz and ligh-cone gauge. Finally, in Sec. 7, some concluding remarks close the paper.

2. Brief Overview on Relativistic Particles

In this section we give an overview of the theory of relativistic particles. Our aim is not to provide an extensive review of such theory, but briefly to recall the notation and conventions, which will be strictly necessary in the following sections (for further details see [40,41,42,43]). In particular we will follow notation and conventions from Ref. [43]. In particular, we will take in our analysis $c = 1$.

To perform the description we consider the world-line L embedded into the D-dimensional space-time M of Lorentzian metric $\eta_{\mu\nu} = diag(-1, 1, \ldots, 1)$, $\mu, \nu = 0, 1, \ldots, D-1$. This embedding is defined by the set of functions: $X^\mu = X^\mu(\tau)$, where τ is the coordinate on the world-line L. The dynamics is encoded in the Nambu-Goto action $S_{NG} = \int_L d\tau L_{NG} = -m \int_L d\tau \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau}}$, where m is the mass of the relativistic particle.

Let $e(\tau)$ be a world-line metric on L. The dynamics of the scalar fields X^μ is described

by a classically equivalent action to the Nambu-Goto action by:

$$S_P = \int_L d\tau L_P$$
$$= \int_L d\tau \left[\frac{e^{-1}(\tau)}{2} \eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} - \frac{e}{2} m^2 \right]. \tag{2.1}$$

While the constraint $T = \frac{\delta S_P}{\delta e} = 0$ is given by

$$T = \frac{e^{-2}}{2} \eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} + \frac{m^2}{2} = 0. \tag{2.2}$$

In the conformal gauge $e(\tau) = \frac{1}{m^2}$ then the action is:

$$S_P = \int_L d\tau \left[\frac{m^2}{2} \eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} - \frac{1}{2} \right]. \tag{2.3}$$

Then, the equations of motion of the free particle are the usual ones

$$\frac{d^2 X^\mu}{d\tau^2} = 0. \tag{2.4}$$

The general solution of Eqs. (2.4) can be written in the form

$$X^\mu(\tau) = x^\mu + \frac{1}{m^2} p^\mu \tau, \tag{2.5}$$

where x^μ and p^μ are real variables. The canonical momentum Π^μ of X^μ is as usual defined by $\Pi_\mu = \frac{\partial L_P}{\partial \dot{X}^\mu}$. This is given by

$$\Pi^\mu(\tau) = m^2 \dot{X}^\mu = p^\mu. \tag{2.6}$$

These solutions (X^μ, Π^μ) satisfy the standard Poisson brackets

$$\{X^\mu(\tau), \Pi^\nu(\tau)\}_P = \eta^{\mu\nu},$$
$$\{X^\mu(\tau), X^\nu(\tau)\}_P = 0 = \{\Pi^\mu(\tau), \Pi^\nu(\tau)\}_P. \tag{2.7}$$

Poisson brackets for variables $x^\mu,\ p^\mu$ are

$$\{x^\mu, p^\nu\}_P = \eta^{\mu\nu}, \tag{2.8}$$

with the remaining independent Poisson brackets being zero.

The constraint (2.2) is written in the conformal gauge as:

$$T = \frac{1}{2}\big[p^2 + m^2\big] = 0. \tag{2.9}$$

This is a primary first class constraint which generate the τ-reparametrization invariance.

Remember that $\Phi_1 = p^2 + m^2 = 0$ is the constraint imposed by the on-shell condition. In the *light-cone gauge* the constraint equations (2.2) can be easily solved and then eliminated. This gauge will be crucial for the deformation quantization of the relativistic particle in order to identify the relevant phase space where implement this quantization.

First, introduce the light-cone coordinates $X^{\pm} := \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$, and the remaining transverse coordinates $\vec{X}_T = X^j$, and $\vec{\Pi}_T = \Pi^j$, where $j = 1, \ldots, D-2$ are left as before. As $X^+(\tau)$ satisfies the equation of motion (2.2) one can choose the coordinate τ in such a manner that $X^+(\tau) = \frac{1}{m^2}p^+\tau$. Of course, this gauge fixing condition $X^+(\tau) = \frac{1}{m^2}p^+\tau$ is explicitly time-dependent. Later we are going back to describe a procedure to deal such a situation (see [38,39]).

In this gauge we can solve the constraint equations in the sense that p^-, are determined totally in terms of p^+ and p^j. Thus the number of degrees of freedom i.e., the independent dynamical variables of the relativistic particle after imposing constraints and gauge conditions are $(x^-, p^+, \vec{X}_T, \vec{\Pi}_T)$, or, equivalently: $(x^-, p^+, \vec{x}_T, \vec{p}_T)$. Thus we have $D-1$ degrees of freedom. For the Poisson bracket for these variables we have

$$\{x^-, p^+\}_P = -1, \qquad \{X^j(\tau), \Pi^k(\tau)\}_P := \delta^{jk},$$

$$\{X^j(\tau), X^k(\tau)\}_P = 0 = \{\Pi^j(\tau), \Pi^k(\tau)\}_P. \tag{2.10}$$

In general, the total Hamiltonian H_T can be written as the sum of the canonical Hamiltonian H_C plus primary first class constraint $\Phi_1(X^\mu, \Pi_\mu)$ as follows

$$H_T = H_C + \lambda_1 \Phi_1(X^\mu, \Pi_\mu), \tag{2.11}$$

where λ_1 is an arbitrary function (Lagrange multipliers). Remember that H_C is vanishing for every reparametrization invariant system. After imposing the light-cone gauge the

Hamiltonian is given by

$$H_T = \frac{p^+ p^-}{m^2} = \frac{1}{2m^2} \sum_{j=1}^{D-2} (p_j^2 + m^2). \tag{2.12}$$

In the next section we study the problem of deformation quantization more systematically from the point of view of constrained systems.

3. Deformation Quantization of Second Class Constrained Systems

In the present section we will overview the WWM formalism [11,12,13] for the case of general systems with second class constraints [25,26,27]. The description includes a deformation of the Dirac bracket and depends on the local considerations of the physical (or reduced) phase space.

3.1. Preliminaries of Constrained Systems

It is well known that the dynamics of the relativistic particle constitutes an example of a first class constrained system. However it also can be regarded as a second class constrained system, when the gauge conditions are taken into account [34,35]).

In order to implement the quantization one can adopt two possible positions. The first one is simply to impose the constraints and gauge fixing conditions at the classical level, and quantizing only the relevant (physical) degrees of freedom. This has been done previously for the deformation quantization of the bosonic string in the light-cone gauge [22]. The second possibility is to develop the formalism of deformation quantization specifically before imposing constraints and gauge fixing. This formalism was developed recently in Refs. [25,26,27]. In the present section we will overview some material from these constructions. In the next sections we will apply them to the relativistic free particle and the charged particle inside an electromagnetic background and its interaction with a dynamical electromagnetic field.

The phase space before imposing constraints and gauge fixing is given by the so called constrained phase space which we assume of the form: $\mathcal{Z}_P = \{Z^\alpha = (X^\mu, \Pi_\mu) \in \mathbb{R}^{2D}\} =$

$T^*\mathbb{R}^D \cong \mathbb{R}^{2D}$. Here Z^α are the coordinates of the extended phase space: $Z^\alpha = X^\mu$ for $\alpha = 1, \ldots, D$ and $Z^\alpha = \Pi_\mu$ for $\alpha = D+1, \ldots, 2D$. The corresponding symplectic two-form is given by $\omega = \omega_{\alpha\beta} dZ^\alpha \wedge dZ^\beta$. Due Darboux's theorem always is possible to find the symplectic basis where the entries of $\omega_{\alpha\beta}$ are constant and moreover $\omega_{\alpha\beta} = \begin{pmatrix} 0 & \mathbf{1}_{D\times D} \\ -\mathbf{1}_{D\times D} & 0 \end{pmatrix}$. Therefore the symplectic form takes the canonical form $\omega = dX^\mu \wedge d\Pi_\mu$.

From the second class constraints perspective, first class constraints plus gauge fixing conditions can be regarded as a second class constraint system [34,35]. Thus we have a set of purely second class constraints: $\Phi_I(Z^\alpha)$ such that $\det(C_{IJ}) = \det(\{\Phi_I, \Phi_J\}_P) \neq 0$ with $I = 1, \ldots, M$, where M as to be even, i.e., $M = 2m$. For the relativistic particle these two second class constraints are $\Phi_1 = p^2 + m^2$ and $\Phi_2 = X^+ - \frac{1}{m^2} p^+ \tau$.

Therefore a constrained submanifold Σ_I can be associated to each constraint Φ_I. Thus the dynamics of the system is described through the intersection of all constraint submanifolds in the form $\mathcal{Z}_\cap = \Sigma_1 \cap \cdots \cap \Sigma_M$. By a theorem of Maskawa and Nakajima [44], we can identify $\mathcal{Z}_\cap$ with the reduced phase space $\mathcal{Z}_P^\mathcal{R}$. Thus in general the constrained submanifold is $\mathcal{Z}_\cap = \mathcal{Z}_P^\mathcal{R} = \mathbb{R}^{2(D-m)}$. For the case of the relativistic particle we have two second class constraints ($m = 1$) and therefore the reduced phase space is $\mathcal{Z}_P^\mathcal{R} = \mathbb{R}^{2(D-1)}$. Thus the number of physical degrees of freedom is $D-1$, i.e., the transverse degrees of freedom $\vec{X}_T$ plus x^-.

One of the features of these kind of systems is that the matrix generated by the constraints: $\det(\mathbf{C}) = \det(C_{IJ}) \neq 0$ is the main object in the description. Therefore there exist the inverse matrix $C_{IJ}^{-1} := C^{IJ} = -C_{IJ}$. The standard procedure establishes that the Poisson bracket on the constraint manifold has to be changed to the Dirac bracket in the extended phase space

$$\{f, g\}_D = \{f, g\}_P - \sum_{I,J=1}^{M} \{f, \Phi_I\}_P C_{IJ}^{-1} \{\Phi_J, g\}_P, \tag{3.1}$$

for any pair of functions $f(Z^\alpha)$ and $g(Z^\alpha)$. With the known property $\{f, g\}_D \overset{\mathcal{Z}_P^\mathcal{R}}{=} \{f, g\}_P$. The Dirac bracket satisfies the same properties as the Poisson bracket.

Similarly as the Poisson bracket, the Dirac bracket also can be expressed in geometrical

terms

$$\omega_D^{-1}(df, dg) = \{f, g\}_D = \omega_D^{\alpha\beta} \frac{\partial f}{\partial Z^\alpha} \cdot \frac{\partial g}{\partial Z^\beta}, \tag{3.2}$$

where we have

$$\omega_D^{\alpha\beta} = \{Z^\alpha, Z^\beta\}_D. \tag{3.3}$$

3.2. Skew-gradient Projection Method

Now locally, in the symplectic basis, one has $\{\Phi_I(Z^\alpha), \Phi_J(Z^\alpha)\}_D = I_{IJ}$ with $I, J = 1, \ldots, 2m$, where $I_{IJ} = \begin{pmatrix} 0 & \mathbf{1}_{m\times m} \\ -\mathbf{1}_{m\times m} & 0 \end{pmatrix}$. Then the restriction over the reduced phase space $\mathcal{Z}_P^{\mathcal{R}}$ starting from the constrained phase space $\mathcal{Z}_P$ can be represented through a skew-gradient projection Z_S of the canonical coordinates Z^α onto $\mathcal{Z}_P^{\mathcal{R}}$ [26,27]

$$Z_S(Z^\alpha) = Z^\alpha + X^I \Phi_I(Z^\alpha) + \frac{1}{2!} X^{IJ} \Phi_I(Z^\alpha)\Phi_J(Z^\alpha) + \cdots. \tag{3.4}$$

Such projection is done according to the condition of having not variations along the phase space flows generated by the constraints Φ_I, i.e.,

$$\{Z_S(Z^\alpha), \Phi_I(Z^\alpha)\}_P = 0, \tag{3.5}$$

which implies that $\Phi_I(Z_S(Z^\alpha)) = 0$. i.e. $Z_S(Z^\alpha)$ lies on the reduced phase space.

This construction has the nice property that any function $f(Z^\alpha)$ on the phase space $\mathcal{Z}_P$ can be projected on $\mathcal{Z}_P^{\mathcal{R}}$ as follows

$$f_S(Z^\alpha) = f(Z_S(Z^\alpha)), \tag{3.6}$$

and satisfies

$$\{f_S(Z^\alpha), \Phi_I(Z^\alpha)\}_P = 0. \tag{3.7}$$

On the reduced phase space we have $Z_S^\alpha(Z^\alpha) = Z^\alpha$ and any observable f can be replaced by f_S. These projected functions satisfy the properties: $\{f, g\}_D = \{f_S, g\}_P = \{f, g_S\}_P = \{f_S, g_S\}_P$. This projection also defines an equivalence class of functions as follows: $[f] = \{f(Z^\alpha) \sim g(Z^\alpha), \text{ iff } f_S(Z^\alpha) \overset{\mathcal{Z}_P^{\mathcal{R}}}{=} g_S(Z^\alpha)\}$. Thus the manner of incorporating this in the

WWM formalism is that the integrations $\int_{\mathcal{Z}_P} F(Z^\alpha) d^{2D}Z$ over the constrained phase space $\mathcal{Z}_P$ should be localized over the space of equivalence classes or simply over the reduced phase space. Therefore, all integrations over the extended phase space should be localized at $\mathcal{Z}_P^{\mathcal{R}}$

$$\int_{\mathcal{Z}_P} \delta(\mathcal{Z}_P^{\mathcal{R}}) F(Z^\alpha) d^{2D}Z = \int_{\mathcal{Z}_P^{\mathcal{R}}} F(Z_S) d^{2(D-m)}Z. \tag{3.8}$$

Later we will go back to discuss this subject.

The dynamics of the systems are controlled by the Dirac bracket and the evolution equation is given by

$$\{f, h\}_D = \frac{\partial f}{\partial t}, \tag{3.9}$$

for a given Hamiltonian function h and for any function f. The projection to the reduced phase space reads

$$\{f, h_S\}_P = \frac{\partial f}{\partial t}, \tag{3.10}$$

with $h_S = h(Z_S)$.

3.3. WWM Formalism in the Constrained Phase Space

The formulation in the extended phase space needs from the action of operators on the extended Hilbert space $\mathcal{H}_e$

$$\widehat{X}^\mu |X^\mu\rangle = X^\mu |X^\mu\rangle, \qquad \widehat{\Pi}^\mu |\Pi^\mu\rangle = \Pi^\mu |\Pi^\mu\rangle,$$

$$[\widehat{X}^\mu, \widehat{\Pi}^\nu] = i\hbar \eta^{\mu\nu}. \tag{3.11}$$

With these definitions we set the normalization of these states as follows

$$\int d^D X |X^\mu\rangle\langle X^\mu| = \widehat{1} \quad \text{and} \quad \int d^D (\frac{\Pi}{2\pi\hbar}) |\Pi^\mu\rangle\langle \Pi^\mu| = \widehat{1}. \tag{3.12}$$

Now we define the *Stratonovich-Weyl (*SW*) quantizer* as follows

$$\widehat{\Omega}(Z^\alpha) = \int d^D\xi \exp\left\{ -\frac{i}{\hbar} \xi^\mu \Pi_\mu \right\} \left| X^\mu - \frac{\xi^\mu}{2} \right\rangle \left\langle X^\mu + \frac{\xi^\mu}{2} \right|$$

$$= \int d^D\left(\frac{\eta}{2\pi\hbar}\right) \exp\left\{ -\frac{i}{\hbar}\eta^\mu X_\mu \right\} \left| \Pi^\mu + \frac{\eta^\mu}{2} \right\rangle \left\langle \Pi^\mu - \frac{\eta^\mu}{2} \right|. \tag{3.13}$$

The SW quantizer has the important properties:

$$\left(\widehat{\Omega}(Z^\alpha)\right)^\dagger = \widehat{\Omega}(Z^\alpha), \qquad \mathrm{Tr}\left(\widehat{\Omega}(Z^\alpha)\right) = 1,$$
$$\mathrm{Tr}\left[\widehat{\Omega}(Z^\alpha)\widehat{\Omega}(Z'^\alpha)\right] = \delta(Z^\alpha - Z'^\alpha). \tag{3.14}$$

Here $\delta(Z^\alpha - Z'^\alpha) = \delta(X^\mu - X'^\mu)\delta\left(\frac{\Pi^\mu - \Pi'^\mu}{2\pi\hbar}\right)$. Here Tr is the trace over the extended Hilbert space $\mathcal{H}_e$.

In the Hilbert space representation the quantization condition reads $\{\cdot,\cdot\}_D \to \frac{1}{i\hbar}[\cdot,\cdot]$. Then we have $[\widehat{\Phi}_I(Z^\alpha), \widehat{\Phi}_J(Z^\alpha)] = i\hbar \widehat{I}_{IJ}$ with $I = 1, \ldots, 2m$. Then the quantum version of the restriction over the reduced phase space $\mathcal{Z}_P^{\mathcal{R}}$ is given by the quantum version of the skew-gradient projection $\widehat{Z}_S$

$$\widehat{Z}_S(\widehat{Z}^\alpha) = \widehat{Z}^\alpha + X^I \widehat{\Phi}_I(\widehat{Z}^\alpha) + \frac{1}{2} X^{IJ} \widehat{\Phi}_I(\widehat{Z}^\alpha)\widehat{\Phi}_J(\widehat{Z}^\alpha) + \cdots. \tag{3.15}$$

Such a projection is also done respecting the condition of having not variations along the phase flows generated by the operator constraints $\widehat{\Phi}_I$, i.e.,

$$[\widehat{Z}_S, \widehat{\Phi}_I(\widehat{Z}^\alpha)] = 0, \tag{3.16}$$

which implies that $\widehat{\Phi}_I(\widehat{Z}_S(Z^\alpha)) = 0$. i.e. $\widehat{Z}_S(\widehat{Z}^\alpha)$ lies on the quantum reduced phase space.

In this construction one has that any operator-valued function $\widehat{f}(\widehat{Z}^\alpha)$ on the phase space is projected on the reduced phase space as follows

$$\widehat{f}_S(\widehat{Z}^\alpha) = \widehat{f}(\widehat{Z}_S(Z^\alpha)), \tag{3.17}$$

and satisfies

$$[\widehat{f}(\widehat{Z}_S(Z^\alpha)), \widehat{\Phi}_I(Z^\alpha)] = 0. \tag{3.18}$$

On the reduced phase space we have $\widehat{Z}_S^\alpha(Z^\alpha) = \widehat{Z}^\alpha$ and therefore any observable $\widehat{f}$ can be replaced $\widehat{f}_S$. These projected operators satisfy the following properties: $[\widehat{f}, \widehat{g}] = [\widehat{f}_S, \widehat{g}] = [\widehat{f}, \widehat{g}_S] = [\widehat{f}_S, \widehat{g}_S]$. This projection also defines an equivalence class of operators

$$[\widehat{f}] = \{\widehat{f}(Z^\alpha) \sim \widehat{g}(Z^\alpha), \;\; \text{iff} \;\; \widehat{f}_S(Z^\alpha) \overset{\mathcal{Z}_P^{\mathcal{R}}}{=} \widehat{g}_S(Z^\alpha)\}. \tag{3.19}$$

The quantum dynamics is described by the evolution equation

$$[\widehat{f}, \widehat{h}_S] = i\hbar \frac{d}{dt}\widehat{f}, \tag{3.20}$$

where $\widehat{h}_S$ is the skew-gradient projection of $\widehat{h}$.

As we will see, Weyl correspondence is then carried over equivalence classes such that it identifies an equivalence class of functions with the corresponding equivalence class of operators. This is a one-one correspondence.

Let $F = F(Z^\alpha)$ be a function on the extended phase space $\mathcal{Z}_P$[4]. Then according to the Weyl rule [11] one assign the following operator $\widehat{F}$ corresponding to the equivalence class of symbols: $[F] = \{F(Z^\alpha) \sim F'(Z^\alpha),\ F_S(Z^\alpha) \stackrel{\mathcal{Z}_P^{\mathcal{R}}}{=} F'_S(Z^\alpha)\}$. Thus we have

$$\widehat{F} = W(F) = \int\ d^{2D}Z\sqrt{\det \mathbf{C}}\prod_{I=1}^{2m}\delta[\Phi_I(Z^\alpha)]\ F(Z^\alpha)\widehat{\Omega}(Z^\alpha), \tag{3.21}$$

where $d^{2D}Z = d^DXd^D(\frac{\Pi}{2\pi\hbar})$. F is called the *symbol* of the operator $\widehat{F}$, $Sym(\widehat{F}) = F(Z^\alpha)$.

After the constraints are imposed through the integration over the δ's in Eq. (3.21) we have

$$\widehat{F} = W(F) = \int\ d^{2(D-m)}Z\sqrt{\det \mathbf{C}}\ F_S(Z^\alpha)\widehat{\Omega}_S(Z^\alpha), \tag{3.22}$$

where $\widehat{\Omega}_S(Z^\alpha)$ is the Stratonovich-Weyl quantizer restricted to $\mathcal{Z}_P^{\mathcal{R}}$. Also we have $\widehat{\Omega}(Z^\alpha)$ and taking the trace one has

$$F(Z^\alpha) = W^{-1}(\widehat{F}) = \mathrm{Tr}\Big(\widehat{\Omega}(Z^\alpha)\widehat{F}\Big). \tag{3.23}$$

Now, let $F = F(Z^\alpha)$ and $G = G(Z^\alpha)$ be elements of $C^\infty(\mathcal{Z}_P)[[\hbar]]$ and let $\widehat{F} = W(F)$ and $\widehat{G} = W(G)$ be their corresponding operators. We would like to find what function on $\mathcal{Z}_P$ corresponds to the product $\widehat{F}\widehat{G}$. This function is denoted by $F \star_D G$ and it is called the *Moyal $*$-product of* F and G.

By using the properties from Eq. (3.23) one gets

[4] Actually F is a formal series in $\hbar$ such that $F(Z^\alpha, \hbar)$ is an element of $C^\infty(\mathcal{Z}_P)[[\hbar]]$, the ring of formal series in $\hbar$ with values in the smooth functions over $\mathcal{Z}_P$.

$$\big(F \star_D G\big)(Z^\alpha) = W^{-1}(\widehat{F}\widehat{G}) = \mathrm{Tr}\big[\widehat{\Omega}(Z^\alpha)\widehat{F}\widehat{G}\big]. \tag{3.24}$$

Substituting Eq. (3.21) into (3.24), using then (3.13) and performing straightforward but tedious manipulations we finally obtain

$$\big(F \star_D G\big)(Z^\alpha) = F(Z_1^\alpha) \exp\left\{ \frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}}_D \right\} G(Z_2^\alpha)\Big|_{Z_1^\alpha = Z_2^\alpha = Z^\alpha}, \tag{3.25}$$

where

$$\overleftrightarrow{\mathcal{P}}_D = \omega_D^{\alpha\beta} \frac{\overleftarrow{\partial}}{\partial Z^\alpha} \cdot \frac{\overrightarrow{\partial}}{\partial Z^\beta}. \tag{3.26}$$

The Moyal bracket associated to the Dirac bracket is given by

$$\{F, G\}_M^{\star_D} = \frac{1}{i\hbar}(F \star_D G - G \star_D F). \tag{3.27}$$

Thus, it constitutes a quantum deformation of the Dirac bracket. This can be regarded also as an extension of the Moyal bracket to second class constrained systems. Thus we have

$$\omega_D^{\alpha\beta} = \{Z^\alpha, Z^\beta\}_M^{\star_D}. \tag{3.28}$$

The deformation consist in the fact that the symbol $F(Z^\alpha)$ is actually a function on $\hbar$ such that $\lim_{\hbar\to 0} F(Z^\alpha;\hbar) = f(Z^\alpha)$ and the Moyal-Dirac bracket $\lim_{\hbar\to 0}\{F, G\}_M^{\star_D} = \{f, g\}_D$. [Remark: In Refs. [26,27] the deformed Dirac bracket $\{F, G\}_M^{\star_D}$ is usually written as the anti-symmetric part of $\star_D$: $F(Z^\alpha) \wedge G(Z^\alpha)$. A symmetric part also can be defined.]

The skew-projection of this Moyal product on the physical phase space $\mathcal{Z}_P^{\mathcal{R}}$ is given by

$$\big(F \star_D G\big)(Z_S^\alpha) = F(Z_{1S}^\alpha) \exp\left\{ \frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}}_D \right\} G(Z_{2S}^\alpha)\Big|_{Z_{1S}^\alpha = Z_{2S}^\alpha = Z_S^\alpha}, \tag{3.29}$$

where

$$\overleftrightarrow{\mathcal{P}}_D = \omega_D^{\alpha\beta} \frac{\overleftarrow{\partial}}{\partial Z^\alpha} \cdot \frac{\overrightarrow{\partial}}{\partial Z^\beta}\Big|_{Z = Z_S} = \overleftrightarrow{\mathcal{P}}. \tag{3.30}$$

Morover, we know that the projection

$$\omega_D^{\alpha\beta}\Big|_{\mathcal{Z}_P^{\mathcal{R}}} = \{Z_S^\alpha, Z_S^\beta\}_M^{\star_D} = \{Z_S^\alpha, Z_S^\beta\}_M. \tag{3.31}$$

Now it is an easy matter to define the Wigner function. Assume $\widehat{\rho}$ to be the density operator of the quantum state. Then according to the general formula (3.23) the function $\rho(Z^\alpha)$ corresponding to $\widehat{\rho}$ reads

$$\rho(Z^\alpha) = W^{-1}(\widehat{\rho}) = \mathrm{Tr}\Big(\widehat{\Omega}(Z^\alpha)\widehat{\rho}\Big).$$

$$= \int d^D\xi \exp\Big\{-\frac{i}{\hbar}\xi^\mu\Pi_\mu\Big\}\Big\langle X^\mu + \frac{\xi^\mu}{2}\Big|\widehat{\rho}\Big|X^\mu - \frac{\xi^\mu}{2}\Big\rangle. \tag{3.32}$$

Then the *Wigner function* $\rho^W(Z^\alpha)$ is defined by a simple modification of Eq. (3.32). Namely,

$$\rho^W(Z^\alpha) := \int d^D(\frac{\xi}{2\pi\hbar}) \exp\Big\{-\frac{i}{\hbar}\xi^\mu\Pi_\mu\Big\}\Big\langle X^\mu + \frac{\xi^\mu}{2}\Big|\widehat{\rho}\Big|X^\mu - \frac{\xi^\mu}{2}\Big\rangle.$$

In particular for the pure state $\widehat{\rho} = |\Psi\rangle\langle\Psi|$ we get

$$\rho^W(Z^\alpha) = \int d^D(\frac{\xi}{2\pi\hbar}) \exp\Big\{-\frac{i}{\hbar}\xi^\mu\Pi_\mu\Big\}\Psi^*\big(X^\mu - \frac{\xi^\mu}{2}\big)\Psi\big(X^\mu + \frac{\xi^\mu}{2}\big),$$

where $\Psi(X^\mu) = \langle X^\mu|\Psi\rangle$ stands for $|\Psi\rangle$ in the Schrödinger representation.

Given ρ^W one can use Eq. (3.21) to find the corresponding density operator $\widehat{\rho}$

$$\widehat{\rho} = \int d^{2D}Z \prod_{I=1}^{2m} \delta[\Phi_I(Z^\alpha)]\sqrt{\det \mathbf{C}}\ \rho^W(Z^\alpha)\widehat{\Omega}(Z^\alpha). \tag{3.33}$$

Consequently, the average value $\langle\widehat{F}\rangle$ reads

$$\langle\widehat{F}\rangle = \frac{\mathrm{Tr}\big(\widehat{\rho}\widehat{F}\big)}{\mathrm{Tr}\big(\widehat{\rho}\big).}$$

Then

$$\langle\widehat{F}\rangle = \frac{\int d^{2D}Z\ \sqrt{\det \mathbf{C}}\prod_{I=1}^{2m}\delta[\Phi_I(Z^\alpha)]\ \rho^W(Z^\alpha)\ \mathrm{Tr}\big(\widehat{\Omega}(Z^\alpha)\widehat{F}\big)}{\int d^{2D}Z\ \sqrt{\det \mathbf{C}}\prod_{I=1}^{2m}\delta[\Phi_I(Z^\alpha)]\ \rho^W(Z^\alpha)}$$

$$= \frac{\int d^{2D}Z\ \sqrt{\det \mathbf{C}}\prod_{I=1}^{2m}\delta[\Phi_I(Z^\alpha)]\ \rho^W(Z^\alpha)W^{-1}\big(\widehat{F}\big)(Z^\alpha)}{\int d^{2D}Z\ \sqrt{\det \mathbf{C}}\prod_{I=1}^{2m}\delta[\Phi_I(Z^\alpha)]\ \rho^W(Z^\alpha)}$$

$$= \frac{\int d^{2(D-m)}Z\ \sqrt{\det \mathbf{C}}\ \rho_S^W(Z^\alpha)F_S(Z^\alpha)}{\int d^{2(D-m)}Z\ \sqrt{\det \mathbf{C}}\ \rho_S^W(Z^\alpha)}. \tag{3.34}$$

3.4. Correlation Functions

Here we are going to present the simple example definition of the correlation functions within the deformation quantization formalism. Namely, we give the Green (Wightman) functions of any pair of functions $\langle F(Z^\alpha) \star_D G(Z^\beta)\rangle$ as follows

$$\langle F(Z^\alpha) \star_D G(Z^\alpha)\rangle = \frac{\int d^{2D}Z \ \sqrt{\det \mathbf{C}} \ \prod_{I=1}^{2m} \delta[\Phi_I(Z^\alpha)]\rho^W(Z^\alpha)F(Z^\alpha) \star_D G(Z^\alpha)}{\int d^{2D}Z \ \sqrt{\det \mathbf{C}} \prod_{I=1}^{2m} \delta[\Phi_I(Z^\alpha)] \ \rho^W(Z^\alpha)}. \quad (3.35)$$

After integration over the variables involved in the constraints we have

$$\langle F(Z^\alpha) \star_D G(Z^\alpha)\rangle = \frac{\int d^{2(D-m)}Z \ \sqrt{\det \mathbf{C}} \ \rho^W_S(Z^\alpha)F_S(Z^\alpha) \star_D G_S(Z^\alpha)}{\int d^{2(D-m)}Z \ \sqrt{\det \mathbf{C}} \ \rho^W_S(Z^\alpha)}. \quad (3.36)$$

In the next section we will implement these results to quantize the relativistic free particle in the light-cone gauge.

4. Deformation Quantization of the Free Relativistic Particle in the Light-cone Gauge

The purpose of this section we will apply the WWM-formalism of second class constrained systems, surveyed in section 3, to the case of the free relativistic particle. As it is well known in the literature, the relativistic particle in the light-cone gauge (see for instance, [43]) can be quantized through the usual Dirac prescription of quantization for the physical degrees of freedom. We will show that this formalism can be consistently applied to the relativistic particle. Before that we want to describe the procedure to determine the dynamics on a second class constrained system.

4.1. Systems with Time-dependent Constraints

Now we briefly overview the description of second class constrained systems which does depend explicitly on time τ. This is precisely the case for all systems which are τ-reparametrization invariant like the case of the relativistic particle. The canonical Hamiltonian analysis, including the Hamiltonian evolution, is quite involved by this fact and

its implementation is quite non-trivial. To overcome these difficulties a procedure based in geometrical analysis was proposed in Ref. [38] and further applied to the relativistic particle in an arbitrary electromagnetic background in Ref. [39].

In general this gauge fixing problem defines a second class constrained system with two possible time-dependent constraints given by:

$$\Phi_1 = p^2 + m^2 = 0, \qquad \Phi_2 = X^+ - \frac{1}{m^2} p^+ \tau, \qquad (\mathbf{L}) \tag{4.1}$$

for the *light-cone* **L** gauge. The **L** gauge will be crucial for the deformation quantization of the relativistic particle.

This formalism allows to find a set of canonical variables for the physical phase space $\mathcal{Z}_P^{\mathcal{R}}$, whose time evolution can be described by an evolution equation of the type

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H_D\}_D, \tag{4.2}$$

for any F and where the original Hamiltonian is modified in the form

$$H_D = H + K. \tag{4.3}$$

Here K satisfies the following differential equation

$$Y = -\omega_{\alpha\beta} \frac{\partial Z^\alpha}{\partial \tau} \frac{\partial Z^\beta}{\partial \xi^a} d\xi^a = dK. \tag{4.4}$$

In the previous equation $\omega_{\alpha\beta}$ is the symplectic form in $\mathcal{Z}_P$ and ξ^a are the local coordinates in the physical phase space $\mathcal{Z}_P^{\mathcal{R}}$.

From this formalism it can be obtained the natural choice of the physical coordinates, found in Sec. 2, is precisely: $\xi^a = (x^-, p^+, \vec{x}_T, \vec{p}_T)$. These physical coordinates satisfy the following Dirac brackets:

$$\{x^-, p_-\}_D = 1, \qquad \{x^j, p_j\}_D = \{X^j(\tau), \Pi_j(\tau)\}_D = \delta^i_j,$$

where $p_- = -p^+$, and $p_j = p^j$. If one solves the differential equation (4.4), we find that $Y = dK = -\frac{1}{m^2} p^+ dp_+$ and therefore $K = \int Y = \frac{1}{m^2} p^+ p^-$. Moreover, the Hamiltonian H_D is given

$$H_D = \frac{1}{m^2} p^+ p^- = \sum_{j=1}^{D-2} \frac{p_j^2 + m^2}{2m^2}. \tag{4.5}$$

Obviously, this expression coincides with the total Hamiltonian in Eq. (2.12).

4.2. WWM Formalism

In the previous section we saw as the reduction to the physical phase space keeping invariant the Hamiltonian dynamics is justified by the formalism, at least locally in the phase space. This is precisely what we need in order to apply the skew-gradient projection formalism revised in Sec. 3 (see Eqs. (3.9) and (3.10)).

According to Sec. 2 and 3, the reduced phase space $\mathcal{Z}_P^{\mathcal{R}}$ of the free relativistic particle in the light-cone gauge is described by the (independent) variables $(x^-, p^+, \vec{X}, \vec{\Pi}_T)$ or $(x^-, p^+, \vec{x}_T, \vec{p}_T)$. Thus this reduced space $\mathcal{Z}_P^{\mathcal{R}} = \mathbb{R}^{2(D-1)}$ is endowed with the symplectic two-form

$$\omega_{\mathcal{R}} = dp_- \wedge dx^- + \sum_{j=1}^{D-2} dp_j \wedge dx^j. \tag{4.6}$$

This is precisely the pull-back of the symplectic form ω of extended phase space, i.e., $\omega_{\mathcal{R}} = \phi^*(\omega)$. Here ϕ of the embedding of the submanifold $\mathcal{Z}_P^{\mathcal{R}}$ into the unconstrained phase space $\mathcal{Z}_P$.

For the present case we have two constraints (4.1). These constraints are of the second class since the determinant of the $\mathbf{C}$-matrix is non-vanishing, i.e. $\det(C_{IJ}) \neq 0$. Remember that locally it can be written as $C_{IJ} = I_{IJ}$ and therefore $\det(C_{IJ}) = 1$.

Let $F = F(X^\mu, \Pi_\mu)$ be a function on the phase space $\mathcal{Z}_P$. Then according to the Weyl rule (3.21) we assign the following operator $\widehat{F}$ corresponding to F

$$\widehat{F} = W(F) = \int d^D X d^D (\frac{\Pi}{2\pi\hbar}) \sqrt{\det \mathbf{C}} \delta[\Phi_1(X^\mu, \Pi_\mu)] \delta[\Phi_2(X^\mu, \Pi_\mu)] \; F(X^\mu, \Pi_\mu) \widehat{\Omega}(X^\mu, \Pi_\mu),$$

where the measures of the integrals are given by $d^D X = dX^0 \cdots dX^{D-1}$ and $d^D \Pi = d\Pi^0 \cdots d\Pi^{D-1}$.

Integration over the constraints reexpress the Weyl rule in $\mathcal{Z}_P^{\mathcal{R}}$ in the form

$$\widehat{F} = W(F) = \int \frac{dx^- dp^+}{2\pi\hbar} d^{D-2} X_T d^{D-2} (\frac{\Pi_T}{2\pi\hbar}) F(x^-, \vec{X}_T, p^+, \vec{\Pi}_T) \widehat{\Omega}(x^-, \vec{X}_T, p^+, \vec{\Pi}_T), \tag{4.7}$$

where $F(x^-, \vec{X}_T, p^+, \vec{\Pi}_T) = F[(X^\mu, \Pi_\mu)_S] = F_S(X^\mu, \Pi_\mu)$ is the skew-gradient projection of the symbol on the constrained phase space $\mathcal{Z}_P$ and similarly for the SW quantizer $\widehat{\Omega}(x^-, \vec{X}_T, p^+, \vec{\Pi}_T) = \widehat{\Omega}[(X^\mu, \Pi_\mu)_S] = \widehat{\Omega}_S(X^\mu, \Pi_\mu)$. Then Eq. (3.13) reduces to

$$\widehat{\Omega}(x^-, \vec{X}_T, p^+, \vec{\Pi}_T) = \int d\xi^- d^{D-2}\xi_T \exp\left\{ -\frac{i}{\hbar}\big(-\xi^- p^+ + \vec{\xi}_T \cdot \vec{\Pi}_T\big)\right\}$$

$$\times \left| x^- - \frac{\xi^-}{2}, \vec{X}_T - \frac{\vec{\xi}_T}{2}\right\rangle \left\langle \vec{X}_T + \frac{\vec{\xi}_T}{2}, x^- + \frac{\xi^-}{2}\right|$$

$$= \int d(\frac{\eta^+}{2\pi\hbar}) d^{D-2}(\frac{\eta_T}{2\pi\hbar}) \exp\left\{ -\frac{i}{\hbar}\big(-x^-\eta^+ + \vec{\eta}_T \cdot \vec{X}_T\big)\right\} \left| p^+ + \frac{\eta^+}{2}, \vec{\Pi}_T + \frac{\vec{\eta}_T}{2}\right\rangle \left\langle \vec{\Pi}_T - \frac{\vec{\eta}_T}{2}, p^+ - \frac{\eta^+}{2}\right|, \tag{4.8}$$

where $d^{D-2}\xi_T \equiv d\xi^1 \dots d\xi^{D-2}$, $d^{D-2}(\frac{\eta_T}{2\pi\hbar}) \equiv d(\frac{\eta^1}{2\pi\hbar}) \dots d(\frac{\eta^{D-2}}{2\pi\hbar})$, $\vec{\xi}_T \cdot \vec{\Pi}_T \equiv \sum_{j=1}^{D-2} \xi^j \Pi^j$ and $\vec{\eta}_T \cdot \vec{X}_T \equiv \sum_{j=1}^{D-2} \eta^j X^j$. In the above expressions the states in the reduced (physical) Hilbert space $\mathcal{H}_{phys}$ defined in the following way

$$|x^-, \vec{X}_T\rangle := |x^-\rangle \otimes \Big(\bigotimes_{j=1}^{D-2} |X^j\rangle \Big), \qquad |p^+, \vec{\Pi}_T\rangle := |p^+\rangle \otimes \Big(\bigotimes_{j=1}^{D-2} |\Pi^j\rangle \Big).$$

The SW quantizer has the same important properties (3.14) on the reduced phase space:

$$\big(\widehat{\Omega}(x^-, \vec{X}_T, p^+, \vec{\Pi}_T)\big)^\dagger = \widehat{\Omega}(x^-, \vec{X}_T, p^+, \vec{\Pi}_T), \tag{4.9}$$

$$\mathrm{tr}\big(\widehat{\Omega}(x^-, \vec{X}_T, p^+, \vec{\Pi}_T)\big) = 1, \tag{4.10}$$

$$\mathrm{tr}\big(\widehat{\Omega}(x^-, \vec{X}_T, p^+, \vec{\Pi}_T)\widehat{\Omega}(x'^-, \vec{X}'_T, p'^+, \vec{\Pi}'_T)\big)$$

$$= \delta(x^- - x'^-)\delta(\frac{p^+ - p'^+}{2\pi\hbar})\delta[\vec{X}_T - \vec{X}'_T]\delta[\frac{\vec{\Pi}_T - \vec{\Pi}'_T}{2\pi\hbar}], \tag{4.11}$$

where tr is the trace over $\mathcal{H}_{phys}$.

Multiplying Eq. (4.7) by $\widehat{\Omega}(x^-, \vec{X}_T, p^+, \vec{\Pi}_T)$ and taking the trace on the reduced Hilbert space, the reduced version of Eq. (3.23) is given by

$$F(x^-, \vec{X}_T, p^+, \vec{\Pi}_T) = \mathrm{tr}\big[\widehat{\Omega}(x^-, \vec{X}_T, p^+, \vec{\Pi}_T)\widehat{F}\big]. \tag{4.12}$$

Now, let $F = F(x^-, \vec{X}_T, p^+, \vec{\Pi}_T)$ and $G = G(x^-, \vec{X}_T, p^+, \vec{\Pi}_T)$ be elements of $C^\infty(\mathcal{Z}_P^{\mathcal{R}})[[\hbar]]$ and let $\widehat{F} = W(F)$ and $\widehat{G} = W(G)$ be their corresponding operators. The function on $\mathcal{Z}_P^{\mathcal{R}}$ that corresponds to the product $\widehat{F}\widehat{G}$ is denoted by $F \star G$ and takes the form

$$\big(F\star G\big)[x^-,\vec{X}_T,p^+,\vec{\Pi}_T] := W^{-1}(\widehat{F}\widehat{G}) = \mathrm{Tr}\big[\widehat{\Omega}(x^-,\vec{X}_T,p^+,\vec{\Pi}_T)\widehat{F}\widehat{G}\big]. \tag{4.13}$$

Remember from Eq. (3.30) that $\overset{\leftrightarrow}{\mathcal{P}}_D$ on $\mathcal{Z}_P^{\mathcal{R}}$ reduces to $\overset{\leftrightarrow}{\mathcal{P}}$. Substituting Eqs. (4.7) and (4.8) into (4.13) and performing straightforward computations we can reexpress (4.13) as

$$\big(F\star G\big)(x^-,\vec{X}_T,p^+,\vec{\Pi}_T) = F(x^-,\vec{X}_T,p^+,\vec{\Pi}_T)\exp\left\{\frac{i\hbar}{2}\overset{\leftrightarrow}{\mathcal{P}}\right\}G(x^-,\vec{X}_T,p^+,\vec{\Pi}_T), \tag{4.14}$$

where

$$\overset{\leftrightarrow}{\mathcal{P}}=\overset{\leftrightarrow}{\mathcal{P}}_\pm + \overset{\leftrightarrow}{\mathcal{P}}_T$$
$$= \left(\frac{\overleftarrow{\partial}}{\partial p^+}\frac{\overrightarrow{\partial}}{\partial x^-} - \frac{\overleftarrow{\partial}}{\partial x^-}\frac{\overrightarrow{\partial}}{\partial p^+}\right) + \sum_{j=1}^{D-2}\left(\frac{\overleftarrow{\partial}}{\partial X^j}\frac{\overrightarrow{\partial}}{\partial \Pi^j} - \frac{\overleftarrow{\partial}}{\partial \Pi^j}\frac{\overrightarrow{\partial}}{\partial X^j}\right). \tag{4.15}$$

Now we proceed to find the Wigner function. Thus the density operator $\widehat{\rho}$ of the quantum state of a the relativistic particle is associated through Weyl correspondence (4.12) to the the function $\rho(x^-,\vec{X}_T,p^+,\vec{\Pi}_T)$

$$\rho(x^-,\vec{X}_T,p^+,\vec{\Pi}_T) = W^{-1}(\widehat{\rho}) = \mathrm{tr}\Big(\widehat{\Omega}(x^-,\vec{X}_T,p^+,\vec{\Pi}_T)\widehat{\rho}\Big)$$

$$= \int d\xi^- d^{D-2}\xi_T \exp\left\{-\frac{i}{\hbar}\big(-\xi^- p^+ + \vec{\xi}_T\cdot\vec{\Pi}_T\big)\right\}\left\langle \vec{X}_T+\frac{\vec{\xi}_T}{2}, x^-+\frac{\xi^-}{2}\middle|\widehat{\rho}\middle| x^- - \frac{\xi^-}{2}, \vec{X}_T-\frac{\vec{\xi}_T}{2}\right\rangle. \tag{4.16}$$

Then as we have seen the *Wigner function* $\rho^W(x^-,\vec{X}_T,p^+,\vec{\Pi}_T)$ is a slightly modification of Eq. (4.16). It is given by

$$\rho^W(x^-,\vec{X}_T,p^+,\vec{\Pi}_T) := \int d(\frac{\xi^-}{2\pi\hbar})d^{D-2}(\frac{\xi}{2\pi\hbar})\exp\left\{-\frac{i}{\hbar}\big(-\xi^- p^+ + \vec{\xi}_T\cdot\vec{\Pi}_T\big)\right\}$$
$$\times\left\langle \vec{X}_T+\frac{\vec{\xi}_T}{2}, x^-+\frac{\xi^-}{2}\middle|\widehat{\rho}\middle| x^- - \frac{\xi^-}{2}, \vec{X}_T-\frac{\vec{\xi}_T}{2}\right\rangle. \tag{4.17}$$

For the pure state $\widehat{\rho} = |\Psi\rangle\langle\Psi|$ it can be written as

$$\rho^W(x^-,\vec{X}_T,p^+,\vec{\Pi}_T) = \int d(\frac{\xi^-}{2\pi\hbar})d^{D-2}(\frac{\xi_T}{2\pi\hbar})\exp\left\{-\frac{i}{\hbar}\big(-\xi^- p^+ + \vec{\xi}_T\cdot\vec{\Pi}_T\big)\right\}$$

$$\times\Psi^*\big(x^- - \frac{\xi^-}{2}, \vec{X}_T-\frac{\vec{\xi}_T}{2}\big)\Psi\big(x^- + \frac{\xi^-}{2}, \vec{X}_T+\frac{\vec{\xi}_T}{2}\big), \tag{4.18}$$

where $\Psi(x^-,\vec{X}_T)=\langle x^-,\vec{X}_T|\Psi\rangle$.

For completeness we will write down the equations in terms of variables $(x^-,p^+,\vec{x}_T,\vec{p}_T)$ one has

$$\widehat{\Omega}(x^-,\vec{x}_T,p^+,\vec{p}_T)=\int d\xi^- d^{D-2}\xi_T \exp\Big\{-\frac{i}{\hbar}\Big(-\xi^- p^+ +\vec{\xi}_T\cdot\vec{p}_T\Big)\Big\}$$

$$\times\Big|x^- -\frac{\xi^-}{2},\vec{x}_T-\frac{\vec{\xi}_T}{2}\Big\rangle\Big\langle\vec{x}_T+\frac{\vec{\xi}_T}{2},x^- +\frac{\xi^-}{2}\Big|$$

$$=\int d(\frac{\eta^+}{2\pi\hbar})d(\frac{\eta_T}{2\pi\hbar})\ \exp\Big\{-\frac{i}{\hbar}\Big(-x^-\eta^+ +\vec{\eta}_T\cdot\vec{x}_T\Big)\Big\}\Big|p^+ +\frac{\eta^+}{2},\vec{p}_T+\frac{\vec{\eta}_T}{2}\Big\rangle\Big\langle\vec{p}_T-\frac{\vec{\eta}_T}{2},p^+ -\frac{\eta^+}{2}\Big|, \tag{4.19}$$

where $\vec{\xi}_T\cdot\vec{p}_T\equiv\sum_{j=1}^{D-2}\xi^j p^j$, $\vec{\eta}_T\cdot\vec{x}_T\equiv\sum_{j=1}^{D-2}\eta^j x^j$.

Then the Moyal $*$-product in terms of these variables reads as

$$\big(F_1\star F_2\big)(x^-,\vec{x}_T,p^+,\vec{p}_T)=F_1(x^-,\vec{x}_T,p^+,\vec{p}_T)\exp\Big\{\frac{i\hbar}{2}\overleftrightarrow{\mathcal{P}}\Big\}F_2(x^-,\vec{x}_T,p^+,\vec{p}_T), \tag{4.20}$$

with

$$\overleftrightarrow{\mathcal{P}}:=\Big(\frac{\overleftarrow{\partial}}{\partial p^+}\frac{\overrightarrow{\partial}}{\partial x^-}-\frac{\overleftarrow{\partial}}{\partial x^-}\frac{\overrightarrow{\partial}}{\partial p^+}\Big)+\sum_{j=1}^{D-2}\Big(\frac{\overleftarrow{\partial}}{\partial x^j}\frac{\overrightarrow{\partial}}{\partial p^j}-\frac{\overleftarrow{\partial}}{\partial p^j}\frac{\overrightarrow{\partial}}{\partial x^j}\Big). \tag{4.21}$$

Finally, for the Wigner function one obtains

$$\rho^W(x^-,\vec{x}_T,p^+,\vec{p}_T)=\int d(\frac{\xi^-}{2\pi\hbar})d^{D-2}(\frac{\xi_T}{2\pi\hbar})\exp\Big\{-\frac{i}{\hbar}\Big(-\xi^- p^+ +\vec{\xi}_T\cdot\vec{p}_T\Big)\Big\}$$

$$\times\Big\langle\vec{x}_T+\frac{\vec{\xi}_T}{2},x^- +\frac{\xi^-}{2}\Big|\widehat{\rho}\Big|x^- -\frac{\xi^-}{2},\vec{x}_T-\frac{\vec{\xi}_T}{2}\Big\rangle \tag{4.22}$$

and in the case of the pure state we get

$$\rho^W(x^-,\vec{x}_T,p^+,\vec{p}_T)=\int d(\frac{\xi^-}{2\pi\hbar})d^{D-2}(\frac{\xi_T}{2\pi\hbar})\exp\Big\{-\frac{i}{\hbar}\Big(-\xi^- p^+ +\vec{\xi}_T\cdot\vec{p}_T\Big)\Big\}$$

$$\times\Psi^*\big(x^- -\frac{\xi^-}{2},\vec{x}_T-\frac{\vec{\xi}_T}{2}\big)\Psi\big(x^- +\frac{\xi^-}{2},\vec{x}_T+\frac{\vec{\xi}_T}{2}\big). \tag{4.23}$$

The application of the inverse Weyl correspondence (4.7) allows to find the density operator $\widehat{\rho}$ starting from a given Wigner function ρ^W. This is given by

$$\widehat{\rho} = \int dx^- d(\frac{p^+}{2\pi\hbar}) d^{D-2}x_T d^{D-2}(\frac{p_T}{2\pi\hbar})\ \rho^W(x^-, \vec{x}_T, p^+, \vec{p}_T)\widehat{\Omega}(x^-, \vec{x}_T, p^+, \vec{p}_T). \tag{4.24}$$

Thus, the correlation function of the operator $\widehat{F}$ is given by

$$\begin{aligned} \langle \widehat{F} \rangle &= \frac{\mathrm{tr}\big(\widehat{\rho}\widehat{F}\big)}{\mathrm{tr}\big(\widehat{\rho}\big)} \\ &= \frac{\int dx^- d(\frac{p^+}{2\pi\hbar}) d^{D-2}x_T d^{D-2}(\frac{p_T}{2\pi\hbar})\rho^W(x^-, \vec{x}_T, p^+, \vec{p}_T)\ \mathrm{Tr}\big(\widehat{\Omega}(x^-, \vec{x}_T, p^+, \vec{p}_T)\widehat{F}\big)}{\int dx^- d(\frac{p^+}{2\pi\hbar}) d^{D-2}x_T d^{D-2}(\frac{p_T}{2\pi\hbar})\rho^W(x^-, \vec{x}_T, p^+, \vec{p}_T)} \\ &= \frac{\int dx^- d(\frac{p^+}{2\pi\hbar}) d^{D-2}x_T d^{D-2}(\frac{p_T}{2\pi\hbar})\ \rho^W(x^-, \vec{x}_T, p^+, \vec{p}_T) W^{-1}\big(\widehat{F}\big)(x^-, \vec{p}_T)}{\int dx^- d(\frac{p^+}{2\pi\hbar}) d^{D-2}x_T d^{D-2}(\frac{p_T}{2\pi\hbar})\rho^W(x^-, \vec{x}_T, p^+, \vec{p}_T)}. \end{aligned} \tag{4.25}$$

4.3. Wigner Function of the Ground State

The Wigner function ρ_0^W of the free particle is defined by

$$p^j \star \rho_0^W = 0 \quad \text{and} \quad p^+ \star \rho_0^W = 0, \tag{4.26}$$

for $j = 1, \ldots, D-2$.

Then after expanding the star product (4.20) we have

$$p^j \rho_0^W = 0, \qquad p^+ \rho_0^W = 0, \tag{4.27}$$

for $j = 1, \ldots, D-2$. The general real solution of Eq. (4.27) satisfying also Eqs. $(I.6)$ and $(I.7)$ (see appendix I) reads

$$\rho_0^W = C\delta(p^1)\ldots\delta(p^{D-2})\delta(p^+), \tag{4.28}$$

where C is a positive integration constant i.e., $C > 0$.

Observe that ρ_0^W is defined by Eqs. (I.6), (I.7) and (4.26) uniquely up to an arbitrary real positive constant factor $C > 0$. This fact can be interpreted in deformation quantization formalism as the uniqueness of the ground state.

4.4. Green Functions

In the present subsection we are going to compute the correlation functions in the context of the deformation quantization formalism. We would like to find the Green (Wightman) functions with $F(Z^\alpha) = X^j(\tau)$ and $G(Z^\alpha) = X^k(\tau')$. Then we will compute from (3.36) $\langle X^j(\tau) \star X^k(\tau')\rangle$. Substituting it into the definition (4.25) and after integration over the constraints we find

$$\langle X^j(\tau) \star X^k(\tau')\rangle = \frac{\int dx^- d(\frac{p^+}{2\pi\hbar}) d^{D-2}x_T d^{D-2}(\frac{p_T}{2\pi\hbar}) \rho_0^W(x^-, \vec{x}_T, p^+, \vec{p}_T) X^j(\tau) \star X^k(\tau')}{\int dx^- d(\frac{p^+}{2\pi\hbar}) d^{D-2}x_T d^{D-2}(\frac{p_T}{2\pi\hbar}) \rho_0^W(x^-, \vec{x}_T, p^+, \vec{p}_T)}. \tag{4.29}$$

where ρ_0^W is given by Eq. (4.28) and $X^j(\tau)$ is given by Eq. (2.5). After some computations we have

$$\langle x^j \star p^k \rangle = \delta_{jk}\frac{i\hbar}{2} = -\langle p^j \star x^k\rangle$$
$$\langle p^j \star p^k\rangle = 0, \qquad \langle x^j \star x^k\rangle = \delta_{jk}\langle x^j x^k\rangle. \tag{4.30}$$

After a bit of algebra one finally find that the two point correlation functions are given by

$$\langle X^j(\tau) \star X^k(\tau')\rangle = \langle x^j x^k\rangle + \frac{i\hbar\delta_{jk}}{2m^2}(\tau' - \tau). \tag{4.31}$$

In the next section we describe the deformation quantization of our particle in a general electromagnetic background. This description will be in the context of the procedure of sections 3 and 4. We will see that the quantization can be carried over in a natural way into this context.

5. Deformation Quantization of a Relativistic Particle in a General Electromagnetic Background

In the present section we will apply the WWM-formalism for second class constraints discussed in section 3, to the case of the relativistic particle in an arbitrary electromagnetic

background. We use also the result to deal time-dependent constraints from symplectic geometrical analysis [39,38]. Thus the particle in an electromagnetic background can be properly canonically quantized in a time-dependent gauge like the temporal gauge or the light-cone gauge. The involved electromagnetic field is non-dynamical and only will works as a general background, thus the number of degrees of freedom do not change and the physical phase space is still our friend $\mathcal{Z}_P^{\mathcal{R}}$. The corresponding action in the conformal gauge is given by

$$S = \int_L d\tau \left[\frac{m^2}{2} \eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau} - \frac{1}{2} \right] - e \int_L d\tau \ A_\mu(X^\rho) \frac{dX^\mu}{d\tau}. \tag{5.1}$$

From this, the canonical momentum is easily computed and it yields

$$P_\mu = m^2 \dot{X}_\mu - eA_\mu = p_\mu - eA_\mu. \tag{5.2}$$

The Poisson bracket of these dynamical variables is given by

$$\{X^\mu, P^\nu\}_P = \eta^{\mu\nu}. \tag{5.3}$$

The presence of an electromagnetic field, as a background, does not modifies the fact that we are dealing with a second class constrained system whose constraints are time-dependent. This leads again to a time-dependent gauge-fixing procedure. Fortunately, as we see in the previous section, there is a formulation explored in Ref. [38,39] for time-dependent constraints.

In general terms, this gauge fixing problem defines a second class constrained system with two time-dependent constraints given by:

$$\Phi_1 = (P + eA)^2 + m^2 = 0, \qquad \Phi_2 = X^+ - \frac{1}{m^2} P^+ \tau, \qquad (\mathbf{L}). \tag{5.4}$$

In Ref. [39], it was shown than even introducing the electromagnetic field as an arbitrary background, there is a Hamiltonian evolution (see, Eq. (4.4)) description on the system on the physical phase space $\mathcal{Z}_P^{\mathcal{R}}$. Also in this case the formalism allows to find a set of canonical variables for the physical phase space $\mathcal{Z}_P^{\mathcal{R}}$ are precisely: $(X^-, P^+, \vec{X}_T, \vec{P}_T)$, whose time evolution can be described by an equation of the type (4.2). Thus, essentially

the number of degrees of freedom are the same than for the free particle. These variables also satisfy the following Dirac brackets

$$\{X^-, P_-\}_D = 1, \qquad \{X^j, P_j\}_D = \{X^j(\tau), \Pi_j(\tau)\}_D = \delta^i_j. \tag{5.5}$$

Moreover, from Eq. (4.4) we can compute $Y = dK = -\frac{1}{m^2}p^+ dp_+$ and therefore $K = \int Y = \frac{1}{m^2}p^+p^-$, with $p^\pm = P^\pm + eA^+$, we have that the total Hamiltonian is

$$\begin{aligned} H_D = K = \int Y &= \frac{1}{m^2}(p^+ + eA^+)(p^- + eA^-) \\ &= \frac{1}{2m^2}\sum_{j=1}^{D-2}\left[(P^j + eA^j)^2 + m^2\right], \end{aligned} \tag{5.6}$$

which is the natural generalization of (2.12).

5.1. Deformation Quantization of the Relativistic Particle in an Arbitrary Electromagnetic Background

In this subsection we are going to describe the deformation quantization for a relativistic particle in an arbitrary electromagnetic background. This is a straightforward generalization of the discussion of subsection 4.2. We have seen that the number of degrees of freedom is exactly the same as the free particle. The reason of that is the fact that electromagnetic field as a background does not introduce new degrees of freedom. Consequently, the Hilbert space description is also the same. Thus the quantization follows straightforward by substituting Poisson brackets by the Dirac one and take the Dirac prescription of quantization. Then we only describe the necessary modifications to quantize by deformation.

Let us consider $F = F[X^\mu, P_\mu]$ be a function on the phase space $\mathcal{Z}_P$. Then according to the Weyl rule (3.21) we assign the following operator $\widehat{F}$ corresponding to F

$$\widehat{F} = W(F) = \int d^DXd^D(\frac{P}{2\pi\hbar})\delta[(P{+}eA)^2{+}m^2]\delta[X^+{-}\frac{1}{m^2}P^+\tau]\sqrt{\det \mathbf{C}}\, F[X^\mu, P_\mu]\widehat{\Omega}[X^\mu, P_\mu]. \tag{5.7}$$

Integration over the irrelevant dof's have essentially the same description as for the free particle described in section 4. Thus we have

$$\widehat{F} = W(F) = \int \frac{dX^- dP^+}{2\pi\hbar} d^{D-2} X_T d^{D-2} (\frac{P_T}{2\pi\hbar}) F[X^-, \vec{X}_T, P^+, \vec{P}_T] \widehat{\Omega}[X^-, \vec{X}_T, P^+, \vec{P}_T], \tag{5.8}$$

where

$F[X^-, \vec{X}_T, P^+, \vec{P}_T]$ and $\widehat{\Omega}[X^-, \vec{X}_T, P^+, \vec{P}_T]$ are the skew-gradient projections on $\mathcal{Z}_P^{\mathcal{R}}$. The projected SW quantizer is given by

$$\widehat{\Omega}[X^-, \vec{X}_T, P^+, \vec{P}_T] = \int d\xi^- d^{D-2} \xi_T \exp\bigg\{ -\frac{i}{\hbar}\big(-\xi^- P^+ + \vec{\xi}_T \cdot \vec{P}_T \big) \bigg\}$$

$$\times \bigg| X^- - \frac{\xi^-}{2}, \vec{X}_T - \frac{\vec{\xi}_T}{2} \bigg\rangle \bigg\langle \vec{X}_T + \frac{\vec{\xi}_T}{2}, x^- + \frac{\xi^-}{2} \bigg|$$

$$= \int d(\frac{\eta^+}{2\pi\hbar}) d^{D-2} (\frac{\eta_T}{2\pi\hbar}) \exp\bigg\{ -\frac{i}{\hbar}\big(-X^- \eta^+ + \vec{\eta}_T \cdot \vec{X}_T \big) \bigg\} \bigg| P^+ + \frac{\eta^+}{2}, \vec{P}_T + \frac{\vec{\eta}_T}{2} \bigg\rangle \bigg\langle \vec{P}_T - \frac{\vec{\eta}_T}{2}, P^+ - \frac{\eta^+}{2} \bigg| \tag{5.9}$$

with the obvious notation $\vec{\xi}_T \cdot \vec{P}_T \equiv \sum_{j=1}^{D-2} \xi^j P^j$ and $\vec{\eta}_T \cdot \vec{X}_T \equiv \sum_{j=1}^{D-2} \eta^j X^j$.

Thus, besides the complications on the gauge fixing procedure, which is encoded in the delta functions from (5.7), otherwise the procedure is identical as the free particle in the light-cone gauge described in section 4 and it will be not repeated here. In the next section we couple the particle with a dynamical electromagnetic field, which will be a more interesting system to study.

6. Deformation Quantization of the Charged Point Particle in a Dynamical Electromagnetic Field

In this section we will consider the deformation quantization for the relativistic point particle interacting with a dynamical electromagnetic field. For simplicity we shall restricted our selves to the case $D = 4$, but, the analysis can be easily extended to higher dimensions. The corresponding action in the conformal gauge is given by

$$S=\int_L d\tau\left[\frac{m^2}{2}\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}-\frac{1}{2}\right]-e\int_L d\tau\, A_\mu(X^\rho)\frac{dX^\mu}{d\tau}+\int d^4x\left\{-\frac{1}{4e^2}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}\zeta(\mathcal{S}[A^\mu])\right\}, \tag{6.1}$$

where ζ is a Lagrange multiplier and the last term in the action is the gauge fixing term. We will consider the Lorentz gauge $G=\partial_\rho A^\rho=0$ and the light-cone gauge $G=A^+=0$.

The momenta are easily computed and them yields

$$P_\mu=\frac{\partial L}{\partial \dot{X}^\mu}=m^2\dot{X}_\mu-eA_\mu=p_\mu-eA_\mu, \qquad \pi^\mu=\frac{\partial L}{\partial(\partial_t A)}=-F^{\mu 0}-\zeta\eta^{\mu 0}G[A^\mu]. \tag{6.2}$$

with $\mu,\nu=0,1,2,3$ and $\pi^i=-E^i$ being the components of the electric field.

Deformation quantization of classical electromagnetic field in the Coulomb gauge was discussed in Ref. [21]. In this section we will adopt rather a covariant description, thus we will employ the general formalism from section 3.

The Poisson brackets for the electromagnetic field is given by

$$\{A_\mu(\vec{x},t),\pi_\nu(\vec{y},t)\}_P=\eta_{\mu\nu}\delta(\vec{x}-\vec{y}),$$

$$\{A_\mu(\vec{x},t),A_\nu(\vec{y},t)\}_P=0=\{\pi_\mu(\vec{x},t),\pi_\nu(\vec{y},t)\}_P. \tag{6.3}$$

The plane-wave expansion of the field variables reads

$$A^\mu(\vec{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3k\left(\frac{\hbar}{2\omega(k)}\right)^{1/2}\left(a^\mu(\vec{k},t)\exp\left(i\vec{k}\cdot\vec{x}\right)+a^{*\mu}(\vec{k},t)\exp\left(-i\vec{k}\cdot\vec{x}\right)\right), \tag{6.4}$$

$$\pi^\mu(\vec{x},t)=\frac{1}{(2\pi)^{3/2}}\int d^3k\; i\left(\frac{\hbar\omega(\vec{k})}{2}\right)^{1/2}\left(a^\mu(\vec{k},t)\exp\left(i\vec{k}\cdot\vec{x}\right)-a^{*\mu}(\vec{k},t)\exp\left(-i\vec{k}\cdot\vec{x}\right)\right), \tag{6.5}$$

where $\omega(\vec{k})=|\vec{k}|$, $a^\mu(\vec{k},t)=\sum_{\lambda=0}^3 a(\vec{k},\lambda,t)\varepsilon^\mu(\vec{k},\lambda)$ and $a(\vec{k},\lambda,t)=a(\vec{k},\lambda)\exp[-i\omega(k)t]$. Notice that the vector potential (6.4) can be decomposed into positive and negative frequency modes: $A^\mu(\vec{x},t)=A^{+\mu}(\vec{x},t)+A^{-\mu}(\vec{x},t)$.

Here $\varepsilon^\mu(\vec{k},\lambda)$ are four polarization vectors which satisfy the following relations

$$\varepsilon^\mu(\vec{k},\lambda)\varepsilon_\mu(\vec{k},\lambda') = \eta_{\lambda\lambda'}, \qquad \sum_{\lambda=0}^{3} \eta_{\lambda\lambda}\varepsilon^\mu(\vec{k},\lambda)\varepsilon^\nu(\vec{k},\lambda) = \eta^{\mu\nu}, \tag{6.6}$$

and

$$k\cdot\varepsilon(\vec{k},1) = k\cdot\varepsilon(\vec{k},2) = 0, \qquad k\cdot\varepsilon(\vec{k},0) = -k\cdot\varepsilon(\vec{k},3). \tag{6.7}$$

The mode functions $a^\mu(\vec{k},\lambda)$ can be obtained from the vector potential and its conjugated variable

$$a^\mu(\vec{k},t) = \frac{1}{(2\pi)^{3/2}(2\hbar\omega(\vec{k}))^{1/2}}\int d^3x \exp\big(-i\vec{k}\cdot\vec{x}\big)\Big(\omega(\vec{k})A^\mu(\vec{x},t) - i\pi^\mu(\vec{x},t)\Big),$$

$$a^{*\mu}(\vec{k},t) = \frac{1}{(2\pi)^{3/2}(2\hbar\omega(\vec{k}))^{1/2}}\int d^3x \exp\big(i\vec{k}\cdot\vec{x}\big)\Big(\omega(\vec{k})A^\mu(\vec{x},t) + i\pi^\mu(\vec{x},t)\Big). \tag{6.8}$$

They satisfy the Poisson brackets

$$\{a_\mu(\vec{k},t), a^*_\nu(\vec{k}',t)\}_P = -\frac{i}{\hbar}\eta_{\mu\nu}\delta(\vec{k}-\vec{k}'),$$

$$\{a_\mu(\vec{k},t), a_\nu(\vec{k}',t)\}_P = 0 = \{a^*_\mu(\vec{k},t), a^*_\nu(\vec{k}',t)\}_P. \tag{6.9}$$

Then in Feynman gauge ($\zeta = 1$) and in the Lorentz gauge the Hamiltonian of the particle and the electromagnetic field reads

$$H_T = H_C + \sum_{I=1}^{2}\lambda_I\Phi_I(X^\mu,P_\mu) - \frac{1}{2}\int d^3x\left(\pi^\mu\pi_\mu - \frac{1}{2}\partial_kA_\mu\partial^kA^\mu\right) + e\int_L d\tau A_\mu(X^\rho)\frac{dX^\mu}{d\tau}$$

$$= H_C + \sum_{I=1}^{2}\lambda_I\Phi_I(X^\mu,P_\mu) + \int d^3k\sum_{\lambda=1}^{2}\hbar\omega(\vec{k})a^*(\vec{k},\lambda)a(\vec{k},\lambda)$$

$$+ \int d^3k\hbar\omega(\vec{k})\big[a^*(\vec{k},3)a(\vec{k},3) - a^*(\vec{k},0)a(\vec{k},0)\big]. \tag{6.10}$$

Notice that this Hamiltonian is not positive-definite and therefore there will be not have a ground state. In order to make this Hamiltonian positive one usually impose the Lorentz gauge condition on the positive frequency modes of the gauge field

$$\partial_\mu A^{+\mu}(\vec{x}, t) = 0. \tag{6.11}$$

Equivalently we have

$$\sum_{\lambda=0}^{3} k \cdot \varepsilon(\vec{k}, \lambda) a(\vec{k}, \lambda) = 0. \tag{6.12}$$

If we use the transversality conditions for the massless photon (6.7), we get

$$\left[a(\vec{k}, 3) - a(\vec{k}, 0)\right] = 0. \tag{6.13}$$

It is easy to see that this condition removes the ambiguity in the Hamiltonian (6.10).

6.1. The Stratonovich-Weyl Quantizer for the Complete System

Consider now the Weyl quantization of the complete system. To this end we deal with the relevant fields at the time $t = 0$. Let $F = F[X^\mu, P_\mu, A^\mu, \pi_\mu]$ be an element of $C^\infty(\mathcal{Z}_P \times \mathcal{Z}_M)[[\hbar]]$. The Weyl rule assign to the functional F the following operator $\widehat{F}$

$$\widehat{F} = W(F[X^\mu, P_\mu, A^\mu, \pi_\mu]) := \int d^4X d^4(\frac{P}{2\pi\hbar})\mathcal{D}A\mathcal{D}(\frac{\pi}{2\pi\hbar})\ \sqrt{\det \mathbf{C}}$$

$$\times\delta[\Phi_1(X^\mu, P_\mu)]\delta[\Phi_2(X^\mu, P_\mu)]\ \delta[G(A^\mu)]\ F[X^\mu, P_\mu, A^\mu, \pi_\mu]\widehat{\Omega}[X^\mu, P_\mu, A^\mu, \pi_\mu], \tag{6.14}$$

where $G[A^\mu] = 0$ represents the Lorentz or light-cone gauge, and $\widehat{\Omega}[X^\mu, P_\mu, A^\mu, \pi_\mu]$ is the Stratonovich-Weyl quantizer for the whole system.

6.2. Lorentz Gauge

First we consider as field variables of the electromagnetic field, the oscillator variables a and a^*. The Weyl rule reads

$$\widehat{F} = W(F[X^\mu, P_\mu, a^\mu, a^{*\mu}]) := \int d^4X d^4(\frac{P}{2\pi\hbar})\mathcal{D}a\mathcal{D}a^*\ \sqrt{\det \mathbf{C}}$$

$$\times\delta[\Phi_1(X^\mu, P_\mu)]\delta[\Phi_2(X^\mu, P_\mu)]\ \delta[G(a^\mu, a^{*\mu})]\ F[X^\mu, P_\mu, a^\mu, a^{*\mu}]\widehat{\Omega}[X^\mu, P_\mu, a^\mu, a^{*\mu}]. \tag{6.15}$$

In particular in the light-cone gauge for the particle sector and the Lorentz gauge (6.13), for the electromagnetic sector we have

$$\widehat{F} = W(F[X^\mu, P_\mu, a^\mu, a^{*\mu}]) := \int d^D X d^D(\frac{P}{2\pi\hbar})\mathcal{D}a\mathcal{D}a^* \ \sqrt{\det \mathbf{C}}$$

$$\times\delta[(P+eA)^2+m^2]\delta[X^+ - \frac{1}{m^2}P^+\tau] \ \delta[a(\vec{k},3)-a(\vec{k},0)]F[X^\mu, P_\mu, a^\mu, a^{*\mu}]\widehat{\Omega}[X^\mu, P_\mu, a^\mu, a^{*\mu}]. \tag{6.16}$$

As we have seen the procedure to gauge fixing the part of particle is easy to implement. The part corresponding to the electromagnetic field is more involved and we will concentrate on it.

The Stratonovich-Weyl quantizer is given by

$$\begin{aligned}\widehat{\Omega}[X^\mu, P_\mu, A^\mu, \pi_\mu] &= \int d^4\xi \mathcal{D}\lambda(\vec{x}) \exp\Big\{ -\frac{i}{\hbar}\xi^\mu P_\mu - \frac{i}{\hbar}\int d^3x \lambda^\mu(\vec{x})\pi_\mu(\vec{x})\Big\} \\ &\quad\times \Big| X^\mu - \frac{\xi^\mu}{2}, A^\mu - \frac{\lambda^\mu}{2}\Big\rangle\Big\langle A^\mu + \frac{\lambda^\mu}{2}, X^\mu + \frac{\xi^\mu}{2}\Big| \\ &= \int d^4(\frac{\eta}{2\pi\hbar})\mathcal{D}\lambda^{*\mu}(\vec{x}) \exp\Big\{ -\frac{i}{\hbar}\eta^\mu X_\mu - \frac{i}{\hbar}\int d^3x \lambda^{*\mu}(\vec{x})A_\mu(\vec{x})\Big\} \\ &\quad\times \Big| P^\mu + \frac{\eta^\mu}{2}, \pi^\mu - \frac{\lambda^{*\mu}}{2}\Big\rangle\Big\langle \pi^\mu + \frac{\lambda^{\mu *}}{2}, P^\mu - \frac{\eta^\mu}{2}\Big|. \end{aligned} \tag{6.17}$$

The structure of the extended Fock space $\mathcal{H}_p \otimes \mathcal{F}_M$ is given by

$$|X^\mu, A^\mu\rangle = |X^\mu\rangle \otimes |A^\mu\rangle$$

where $|X^\mu\rangle = |X^+, X^-, \vec{X}_T\rangle = |X^+\rangle \otimes |X^-\rangle \otimes |\vec{X}_T\rangle$ and $|A^\mu\rangle = |a(\vec{k},0), a(\vec{k},3), \vec{a}_T\rangle = |a(\vec{k},0)\rangle \otimes |a(\vec{k},3)\rangle \otimes |\vec{a}_T\rangle$.

The commutation relations are given by

$$[\widehat{a}^\mu(\vec{k},t), \widehat{a}^{*\nu}(\vec{k}',t)] = \eta^{\mu\nu}\delta(\vec{k}-\vec{k}'),$$

$$[\widehat{a}^\mu(\vec{k},t), \widehat{a}^{\nu}(\vec{k}',t)] = 0 = [\widehat{a}^{*\mu}(\vec{k},t), \widehat{a}^{*\nu}(\vec{k}',t)]. \tag{6.18}$$

Following a similar analysis to the description of the supersymmetric Weyl correspondence from Ref. [16], we get

$$\widehat{\Omega}[X^\mu, P_\mu, a, a^*] = \widehat{\Omega}[X^\mu, P_\mu] \otimes \widehat{\Omega}[a, a^*]. \tag{6.19}$$

These operators, of course, can be skew-gradient projected as follows

$$\widehat{\Omega}_S[X^\mu, P_\mu, a, a^*] = \widehat{\Omega}_S[X^\mu, P_\mu] \otimes \widehat{\Omega}_S[a, a^*]. \tag{6.20}$$

The Star-Product

The Moyal $\star$-product in the complete system can be constructed in a similar way as for the free case. Let $F[X^\mu, P_\mu, a, a^*]$ and $G[X^\mu, P_\mu, a, a^*]$ be two functionals on $\mathcal{Z}_P \times \mathcal{Z}_M$ and let $\widehat{F}$ and $\widehat{G}$ be their corresponding Weyl operators, then

$$\big(F \star G\big)[X^\mu, P_\mu, a, a^*] = F[X^\mu, P_\mu, a, a^*] \exp\left(\frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}}_{pM}\right) G[X^\mu, P_\mu, a, a^*], \tag{6.21}$$

where

$$\overleftrightarrow{\mathcal{P}}_{pM} = \overleftrightarrow{\mathcal{P}}_p + \overleftrightarrow{\mathcal{P}}_M \ . \tag{6.22}$$

Here

$$\overleftrightarrow{\mathcal{P}}_p := \sum_{\mu=0}^{3} \left(\frac{\overleftarrow{\partial}}{\partial X^\mu} \frac{\overrightarrow{\partial}}{\partial P_\mu} - \frac{\overleftarrow{\partial}}{\partial P^\mu} \frac{\overrightarrow{\partial}}{\partial X_\mu} \right) \tag{6.23}$$

and

$$\begin{aligned}
\overleftrightarrow{\mathcal{P}}_M := & \sum_{\lambda=0}^{3} \int d^3k \left(\frac{\overleftarrow{\delta}}{\delta a(\vec{k}, \lambda)} \frac{\overrightarrow{\delta}}{\delta a^*(\vec{k}, \lambda)} - \frac{\overleftarrow{\delta}}{\delta \ a^*(\vec{k}, \lambda)} \frac{\overrightarrow{\delta}}{\delta a(\vec{k}, \lambda)} \right) \\
= & \sum_{\lambda=1}^{2} \int d^3k \left(\frac{\overleftarrow{\delta}}{\delta a(\vec{k}, \lambda)} \frac{\overrightarrow{\delta}}{\delta a^*(\vec{k}, \lambda)} - \frac{\overleftarrow{\delta}}{\delta \ a^*(\vec{k}, \lambda)} \frac{\overrightarrow{\delta}}{\delta a(\vec{k}, \lambda)} \right) \\
& + \int d^3k \left(\frac{\overleftarrow{\delta}}{\delta a(\vec{k}, 0)} \frac{\overrightarrow{\delta}}{\delta a^*(\vec{k}, 0)} - \frac{\overleftarrow{\delta}}{\delta \ a^*(\vec{k}, 0)} \frac{\overrightarrow{\delta}}{\delta a(\vec{k}, 0)} \right) \\
& + \int d^3k \left(\frac{\overleftarrow{\delta}}{\delta a(\vec{k}, 3)} \frac{\overrightarrow{\delta}}{\delta a^*(\vec{k}, 3)} - \frac{\overleftarrow{\delta}}{\delta \ a^*(\vec{k}, 3)} \frac{\overrightarrow{\delta}}{\delta a(\vec{k}, 3)} \right).
\end{aligned} \tag{6.24}$$

The Wigner Functional for the Complete System

Now we are going to implement the gauge constraints at the quantum level. These are the particle and the gauge field parts. The particle part is the usual light-cone gauge, while the electromagnetic part will be the quantum version of the Lorentz gauge

$$\partial_\mu \widehat{A}^{+\mu}(\vec{x},t)|\Phi\rangle = 0. \tag{6.25}$$

Equivalently we have

$$\sum_{\lambda=0}^{3} k\cdot\varepsilon(\vec{k},\lambda)\widehat{a}(\vec{k},\lambda)|\Phi\rangle = 0. \tag{6.26}$$

States $|\Phi\rangle$ are factorized as $|\Phi\rangle = |\Phi_P\rangle \otimes |\Phi_U\rangle$, where $|\Phi_P\rangle$ are the physical states and $|\Phi_U\rangle$ are unphysical ones. Eq. (6.26) leads to

$$\big[\widehat{a}(\vec{k},3)-\widehat{a}(\vec{k},0)\big]|\Phi\rangle = \big[\widehat{a}(\vec{k},3)-\widehat{a}(\vec{k},0)\big]|\Phi_U\rangle = 0. \tag{6.27}$$

Here we have used the well known transversality conditions for the massless photon (6.7).

Equivalently one can use the light-cone gauge for the electromagnetic part

$$\widehat{A}^{+}(\vec{x},t)|\Phi\rangle = 0. \tag{6.28}$$

In the procedure of quantization there will be used both of them.

Now we are going to compute the physical Winger functional of the ground state. For the composed system it was shown in Ref. [16], that the Wigner function can be factorized as

$$\rho_0^W(X^\mu,P_\mu,a,a^*) = \rho_0^W(X^\mu,P_\mu)\cdot\rho_0^W(a,a^*). \tag{6.29}$$

The skew-gradient projected is the physical Wigner functional

$$\rho_{S0}^W(X^\mu,P_\mu,a,a^*) = \rho_{S0}^W(X^\mu,P_\mu)\cdot\rho_{S0}^W(a,a^*). \tag{6.30}$$

For the particle case will have $\rho_{S0}^W(X^\mu,P_\mu) = \rho_0^W(X^-,\vec{X}_T,P^+,\vec{P}_T)$ as we got previously (see, Eq. (4.28)). Thus the Wigner functional $\rho_{S0}^W(X^\mu,P_\mu)\cdot\rho_0^W(a,a^*)$ will be a solution of the following systems of equations

$$P^j\star\rho_0^W = 0, \qquad P^+\star\rho_0^W = 0,$$

$$\vec{a}_T(\vec{k},j)\star\rho_0^W=0,\qquad \big[a(\vec{k},3)-a(\vec{k},0)\big]\star\rho_0^W=0. \tag{6.31}$$

Taking $\rho_0^W(a,a^*)=\rho_0^W(P)\cdot\rho_0^W(U)$ and using the Moyal product (6.23) and (6.24) we get

$$\begin{aligned}
&P^j\rho_0^W=0, \qquad P^+\rho_0^W=0,\\
&a_T^j(\vec{k})\rho_0^W(P)+\frac{1}{2}\frac{\delta\rho_0^W(P)}{\delta a_T^{*j}(\vec{k})}=0,\qquad \big[a(\vec{k},3)-a(\vec{k},0)\big]\rho_0^W(U)=0.
\end{aligned} \tag{6.32}$$

The solution is as follows to all these constraints is given by the product of $\rho_0^W(X^-,\vec{X}_T,P^+,\vec{P}_T)=\delta(\vec{P}_T)\delta(P^+)$ and $\rho_0^W(a,a^*)=\rho_0^W(P)\cdot\rho_0^W(U)=\exp\left\{-2\sum_{j=1}^2\int d^3k a_T^{*j}(\vec{k})a_T^j(\vec{k})\right\}\delta\big[a(\vec{k},3)-a(\vec{k},0)\big]$. Then we have

$$\rho_0^W(X^-,\vec{X}_T,P^+,\vec{P}_T,a,a^*)=C\exp\left\{-2\sum_{j=1}^2\int d^3k a_T^{*j}(\vec{k})a_T^j(\vec{k})\right\}\delta\big[a(\vec{k},3)-a(\vec{k},0)\big]\delta(\vec{P}_T)\delta(P^+), \tag{6.33}$$

where $C>0$. This Wigner function composed by the Wigner function of the particle plus that of an infinite set of oscillators resembles very much that of the bosonic string [22].

Thus, once we have the Wigner functional of the ground state, the correlation functions of gauge invariant operators $\widehat{\mathcal{O}}$ can be computed by

$$\begin{aligned}
\langle\widehat{\mathcal{O}}\rangle &= \frac{\mathrm{Tr}\big(\widehat{\rho}_{phys}\widehat{\mathcal{O}}\big)}{\mathrm{Tr}\big(\widehat{\rho}_{phys}\big)}\\
&=\frac{\int dX^-dP^+d^2X_Td^2P_T\mathcal{D}a_T\mathcal{D}a_T^*\rho_0^W[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{a}_T,\vec{a}_T^*]\ \mathrm{Tr}\big(\widehat{\Omega}[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{a}_T,\vec{a}_T^*]\widehat{\mathcal{O}}\big)}{\int dX^-dP^+d^2X_Td^2P_T\rho_0^W[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{a}_T,\vec{a}_T^*]}\\
&=\frac{\int dX^-dP^+d^2X_Td^2P_T\mathcal{D}a_T\mathcal{D}a_T^*\ \rho_0^W[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{a}_T,\vec{a}_T^*]\mathcal{O}_T[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{a}_T,\vec{a}_T^*]}{\int dX^-dP^+d^2X_Td^2P_T\rho_0^W[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{a}_T,\vec{a}_T^*]}.
\end{aligned} \tag{6.34}$$

Gauge Invariant Reduction

After integrating out the spurious degrees of freedom with the uses of (6.24) we have

$$\begin{aligned}
\widehat{F}=W(F)=\int\frac{dX^-dP^+}{2\pi\hbar}d^2X_Td^2(\frac{P_T}{2\pi\hbar})\mathcal{D}a_T\mathcal{D}a_T^*\ \sqrt{\det\mathbf{C}}\\
\times F[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{a}_T,\vec{a}_T^*]\widehat{\Omega}[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{a}_T,\vec{a}_T^*]
\end{aligned} \tag{6.35}$$

where $F[X^-, \vec{X}_T, P^+, \vec{P}_T, \vec{a}_T, \vec{a}^*_T] = F[[X^\mu, P_\mu, a^\mu, a^*_\mu]_S] = F_S[X^\mu, P_\mu, a^\mu, a^*_\mu]$ is the skew-gradient projected symbol on the reduced phase space $\mathcal{Z}^{\mathcal{R}}_P \times \mathcal{Z}^{\mathcal{R}}_M$ and similarly for the Stratonovich-Weyl quantizer $\widehat{\Omega}[X^-, \vec{X}_T, P^+, \vec{P}_T, \vec{a}_T, \vec{a}^*_T] = \widehat{\Omega}[[X^\mu, P_\mu, a^\mu, a^*_\mu]_S] = \widehat{\Omega}_S[X^\mu, P_\mu, a^\mu, a^*_\mu]$ which is given by

$$\widehat{\Omega}[X^-, \vec{X}_T, P^+, \vec{P}_T, \vec{a}_T, \vec{a}^*_T] = \widehat{\Omega}[X^-, \vec{X}_T, P^+, \vec{P}_T] \otimes \widehat{\Omega}[\vec{a}_T, \vec{a}^*_T]$$

$$= \int d\xi^- d^2\xi_T \mathcal{D}\lambda_T \exp\bigg\{ -\frac{i}{\hbar}\big(-\xi^- P^+ + \vec{\xi}_T \cdot \vec{P}_T\big) - \frac{i}{\hbar}\int d^3x \big(\vec{\lambda}_T \cdot \vec{a}^*_T\big)\bigg\}$$

$$\times \bigg| X^- - \frac{\xi^-}{2}, \vec{X}_T - \frac{\vec{\xi}_T}{2}, \vec{a}_T - \frac{\vec{\lambda}_T}{2}\bigg\rangle\bigg\langle \vec{a}_T + \frac{\vec{\lambda}_T}{2}, \vec{X}_T + \frac{\vec{\xi}_T}{2}, X^- + \frac{\xi^-}{2}\bigg|$$

$$= \int d(\frac{\eta^+}{2\pi\hbar}) d^2(\frac{\eta_T}{2\pi\hbar}) \mathcal{D}\lambda^*_T \exp\bigg\{ -\frac{i}{\hbar}\big(-X^-\eta^+ + \vec{\eta}_T \cdot \vec{X}_T\big) - \frac{i}{\hbar}\int d^3x \big(\vec{\lambda}^*_T \cdot \vec{a}_T\big)\bigg\}$$

$$\times \bigg| P^+ + \frac{\eta^+}{2}, \vec{P}_T + \frac{\vec{\eta}_T}{2}, \vec{a}^*_T - \frac{\vec{\lambda}^*_T}{2}\bigg\rangle\bigg\langle \vec{a}^*_T + \frac{\vec{\lambda}^*_T}{2}, \vec{P}_T - \frac{\vec{\eta}_T}{2}, P^+ - \frac{\eta^+}{2}\bigg| \qquad (6.36)$$

with the obvious notation $\vec{\xi}_T \cdot \vec{P}_T \equiv \sum_{j=1}^2 \xi^j P^j$, $\vec{\eta}_T \cdot \vec{X}_T \equiv \sum_{j=1}^2 \eta^j X^j$, $\vec{\lambda}_T \cdot \vec{a}^*_T \equiv \sum_{j=1}^2 \lambda^j a^{*j}$ and $\vec{\lambda}^*_T \cdot \vec{a}_T \equiv \sum_{j=1}^2 \lambda^{*j} a^j$.

6.3. Light-cone Gauge

Now we move from the field variables to the light-cone variables. In particular in the light-cone gauge for both the particle and the Maxwell gauge field. The gauge field A^μ can be decomposed in $(A^+, A^-, \vec{A}_T)$. It will be more convenient to work in the moment space representation $(A^+(p), A^-(p), \vec{A}_T)$. We will work in the light-cone gauge $A^+(p) = 0$. In this gauge fields $A^-(p)$ are determined in terms of p^+ and the transverse components $\vec{A}_T$ i.e., $A^-(p) = \frac{1}{P^+}\vec{p}_T \cdot \vec{A}_T$. Thus for $p^2 = 0$ the gauge field is completely determined by the two transverse degrees of freedom $\vec{A}_T$. The light-cone gauge can be incorporated into the Weyl correspondence as follows

$$\widehat{F} = W(F[X^\mu, P_\mu, A^\mu, \pi_\mu]) := \int d^D X d^D(\frac{P}{2\pi\hbar}) \mathcal{D}A \mathcal{D}(\frac{\pi}{2\pi\hbar}) \sqrt{\det \mathbf{C}}$$

$$\times \delta[(P+eA)^2 + m^2] \delta[X^+ - \frac{1}{m^2}P^+\tau]\ \delta[A^+] F[X^\mu, P_\mu, A^\mu, \pi_\mu] \widehat{\Omega}[X^\mu, P_\mu, A^\mu, \pi_\mu]. \quad (6.37)$$

The Star-Product

We start with the functionals $F[X^\mu, P_\mu, A^\mu, \pi_\mu]$ and $G[X^\mu, P_\mu, A^\mu, \pi_\mu]$ defined on $\mathcal{Z}_P \times \mathcal{Z}_M$ and let $\widehat{F}$ and $\widehat{G}$ be their corresponding operators, then

$$\begin{aligned}&\big(F \star G\big)[X^\mu, P_\mu, A^\mu, \pi_\mu]\\ &= F[X^\mu, P_\mu, A^\mu, \pi_\mu] \exp\left(\frac{i\hbar}{2} \overleftrightarrow{\mathcal{P}}_{pM}\right) G[X^\mu, P_\mu, A^\mu, \pi_\mu],\end{aligned} \tag{6.38}$$

$$\overleftrightarrow{\mathcal{P}}_{pM} = \overleftrightarrow{\mathcal{P}}_p + \overleftrightarrow{\mathcal{P}}_M, \tag{6.39}$$

where

$$\overleftrightarrow{\mathcal{P}}_p := \sum_{\mu=0}^{3} \left(\frac{\overleftarrow{\partial}}{\partial X^\mu} \frac{\overrightarrow{\partial}}{\partial P_\mu} - \frac{\overleftarrow{\partial}}{\partial P^\mu} \frac{\overrightarrow{\partial}}{\partial X_\mu} \right), \tag{6.40}$$

and

$$\begin{aligned}\overleftrightarrow{\mathcal{P}}_M &:= \sum_{\mu=0}^{3} \int d^3x \left(\frac{\overleftarrow{\delta}}{\delta A^\mu(\vec{x})} \frac{\overrightarrow{\delta}}{\delta \pi_\mu(\vec{x})} - \frac{\overleftarrow{\delta}}{\delta\ \pi^\mu(\vec{x})} \frac{\overrightarrow{\delta}}{\delta A_\mu(\vec{x})} \right)\\ &= \int d^3x \left(\frac{\overleftarrow{\delta}}{\delta A^+(\vec{x})} \frac{\overrightarrow{\delta}}{\delta \pi_-(\vec{x})} - \frac{\overleftarrow{\delta}}{\delta\ \pi^+(\vec{x})} \frac{\overrightarrow{\delta}}{\delta A_-(\vec{x})} \right) + \sum_{j=1}^{2} \int d^3x \left(\frac{\overleftarrow{\delta}}{\delta A^j(\vec{x})} \frac{\overrightarrow{\delta}}{\delta \pi_j(\vec{x})} - \frac{\overleftarrow{\delta}}{\delta\ \pi^j(\vec{x})} \frac{\overrightarrow{\delta}}{\delta A_j(\vec{x})} \right).\end{aligned} \tag{6.41}$$

The Wigner Functional for the Complete System

Now we are going to compute the Winger functional of the ground state. The Wigner functional we want to compute is

$$\rho_0^W(X^\mu, P_\mu, a, a^*) = \rho_{S0}^W(X^\mu, P_\mu) \cdot \rho_0^W(a, a^*). \tag{6.42}$$

This is defined by the conditions

$$\begin{aligned} P^j \star \rho_0^W &= 0, & P^+ \star \rho_0^W &= 0,\\ \vec{a}_T(\vec{p}, j) \star \rho_0^W &= 0, & A^+(\vec{p}) \star \rho_0^W &= 0. \end{aligned} \tag{6.43}$$

As before, take $\rho_0^W(a,a^*) = \rho_0^W(P)\otimes\rho_0^W(U)$ and with aid of the Moyal product (6.40) and (6.41) we get

$$P^j\rho_0^W = 0, \qquad P^+\rho_0^W = 0,$$

$$a_T^j(\vec{p})\rho_0^W(P) + \frac{1}{2}\frac{\delta\rho_0^W(P)}{\delta a_T^{*j}(\vec{p})} = 0, \qquad A^+(\vec{p})\rho_0^W(U) = 0. \tag{6.44}$$

The solution is as follows to all these constraints is given by

$$\rho_0^W(X^-,\vec{X}_T,P^+,\vec{P}_T,a,a^*) = C\exp\Big\{-2\sum_{j=1}^{2}\int d^3p a_T^{*j}(\vec{p})a_T^j(\vec{p})\Big\}\delta\big[A^+(\vec{p})\big]\delta(\vec{P}_T)\delta(P^+), \tag{6.45}$$

with $C>0$. The correlation functions also can be computed in this gauge with this Wigner function.

Gauge Invariant Reduction

After integrating out the spurious degrees of freedom it yields

$$\widehat{F} = W(F) = \int\frac{dX^-dP^+}{2\pi\hbar}d^2X_Td^2(\frac{P_T}{2\pi\hbar})\mathcal{D}A_T\mathcal{D}\pi_T\ \sqrt{\det\mathbf{C}}$$

$$\times F[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{A}_T,\vec{\pi}_T]\widehat{\Omega}[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{A}_T,\vec{\pi}_T] \tag{6.46}$$

. The Stratonovich-Weyl quantizer is given by

$$\widehat{\Omega}[X^-,\vec{X}_T,P^+,\vec{P}_T,\vec{A}_T,\vec{\pi}_T] = \widehat{\Omega}[X^-,\vec{X}_T,P^+,\vec{P}_T]\otimes\widehat{\Omega}[\vec{A}_T,\vec{\pi}_T]$$

$$= \int d\xi^-d^2\xi_T\mathcal{D}\lambda_T\exp\Big\{-\frac{i}{\hbar}\big(-\xi^-P^+ + \vec{\xi}_T\cdot\vec{P}_T\big) - \frac{i}{\hbar}\int d^3x\vec{\lambda}_T(\vec{x})\cdot\vec{\pi}_T(\vec{x})\Big\}$$

$$\times\Big|X^- - \frac{\xi^-}{2},\vec{X}_T - \frac{\vec{\xi}_T}{2},\vec{A}_T - \frac{\vec{\lambda}_T}{2}\Big\rangle\Big\langle\vec{A}_T + \frac{\vec{\lambda}_T}{2},\vec{X}_T + \frac{\vec{\xi}_T}{2},X^- + \frac{\xi^-}{2}\Big|$$

$$= \int d(\frac{\eta^+}{2\pi\hbar})d^2(\frac{\eta_T}{2\pi\hbar})\mathcal{D}\lambda_T^*\exp\Big\{-\frac{i}{\hbar}\big(-X^-\eta^+ + \vec{\eta}_T\cdot\vec{X}_T\big) - \frac{i}{\hbar}\int d^3x\vec{\lambda}_T^*(\vec{x})\cdot\vec{A}_T(\vec{x})\Big\}$$

$$\times\Big|P^+ + \frac{\eta^+}{2},\vec{P}_T + \frac{\vec{\eta}_T}{2},\vec{\pi}_T - \frac{\vec{\lambda}_T^*}{2}\Big\rangle\Big\langle\vec{\pi}_T + \frac{\vec{\lambda}_T^*}{2},\vec{P}_T - \frac{\vec{\eta}_T}{2},P^+ - \frac{\eta^+}{2}\Big|. \tag{6.47}$$

7. Final Remarks

In the present paper we have applied the WWM-formalism to quantize the relativistic free particle and the relativistic particle in a general electromagnetic background. We have used recent results concerning the deformation quantization of second class constrained systems [25,26,27]. We have shown that this formalism serves to quantize both kind of systems in a way that resembles the Faddeev and Popov quantization of a gauge theory through Feynman path integrals. It allows to describe the deformation quantization of constrained systems in a more geometric way. This approach also is useful to quantize by deformation the charged particle interacting with a dynamical electromagnetic field in the Lorentz and the light-cone gauges.

Moreover, we obtain that the WWM-formalism used, totally justifies the deformation quantization of the relativistic particle in the light-cone gauge. Which can be regarded also as the low energy limit when the size of the string vanishes, i.e., $\alpha' = \ell_S^2 \to 0$. Thus, our results are consistent with those of [22] in this limit.

We have shown that deformation quantization of the relativistic particle gives the same results as the canonical quantization and path integral methods. Thus, this equivalence constitutes an further evidence of the validity of these proposals [25,26,27] for systems with second class constraints. The Stratonovich-Weyl quantizer, Weyl correspondence, Moyal product and the Wigner function are obtained for all the analyzed systems.

The extension of the formalism to the superparticle described by the supersymmetric action

$$S_{SP} = \int_L d\tau \; \eta_{\mu\nu} \left(\frac{dX^\mu}{d\tau} - i\overline{\theta}\Gamma^\mu \frac{d\theta}{d\tau} \right) \left(\frac{dX^\nu}{d\tau} - i\overline{\theta}\Gamma^\nu \frac{d\theta}{d\tau} \right).$$

and more general systems like superstring theory is one of the open problems that will be pursued in the near future. It is interesting also to apply all these matters to more complicated second class constrained systems as the BRST quantization in gauge theories and Batalin-Vilkovisky quantization. It would be interesting also to describe the deformation quantization of the closed and open strings coupled with Neveu-Schwartz and Ramond-Ramond fields. We hope to address some of these topics in the future.

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Appendix I. Wigner Functions and Pure States

For the pure state the density operator is $\widehat{\rho} = |\Psi\rangle\langle\Psi|$. One can substitute it into Eq. (6.9), and with the aid of Eq. (6.3) after some lengthy algebra we get

$$|\Psi(y^-, \vec{y}_T)|^2 = \int d(\frac{p^+}{2\pi\hbar}) d^{D-2}(\frac{p_T}{2\pi\hbar})\ \rho^W(y^-, \vec{y}_T, p^+, \vec{p}_T). \tag{I.1}$$

Here we assumed that $\Psi(y^-, \vec{y}_T) \neq 0$. From this one can extract the wave function $\Psi(x^-, \vec{x}_T)$ in terms of the corresponding Wigner function ρ^W

$$\Psi(x^-, \vec{x}_T) = \mathcal{N}^{-1} \exp\{i\varphi\} \int d(\frac{p^+}{2\pi\hbar}) d^{D-2}(\frac{p_T}{2\pi\hbar})\ \rho^W(\frac{x^- + y^-}{2}, \frac{\vec{x}_T + \vec{y}_T}{2}, p^+, \vec{p}_T)$$
$$\times \exp\Big\{ -\frac{i}{\hbar}\big(-(x^- - y^-)p^+ + (\vec{x}_T - \vec{y}_T)\cdot \vec{p}_T \big)\Big\}, \tag{I.2}$$

where $\exp(i\varphi)$ is a phase factor with φ being a real constant and

$$\mathcal{N} = \Big(\int d(\frac{p^+}{2\pi\hbar}) d^{D-2}(\frac{p_T}{2\pi\hbar}) \rho^W(y^-, \vec{y}_T, p^+, \vec{p}_T)\Big)^{1/2}. \tag{I.3}$$

In terms of variables $\vec{X}_T(\tau)$ and $\vec{\Pi}_T(\tau)$ one has

$$\Psi(x^-, \vec{X}_T) = \mathcal{N}'^{-1} \exp\{i\varphi\} \int d(\frac{p^+}{2\pi\hbar}) d^{D-2}(\frac{\Pi_T}{2\pi\hbar})\ \rho^W(\frac{x^- + y^-}{2}, \frac{\vec{X}_T + \vec{Y}_T}{2}, p^+, \vec{\Pi}_T)$$
$$\times \exp\Big\{ -\frac{i}{\hbar}\big(-(x^- - y^-)p^+ + (\vec{X}_T(\tau) - \vec{Y}_T(\tau))\cdot \vec{\Pi}_T \big)\Big\}, \tag{I.4}$$

where $\vec{X}_T(\tau)\cdot\vec{\Pi}_T(\tau) \equiv \sum_{j=1}^{D-2} X^j \Pi^j$ and

$$\mathcal{N}' = \Big(\int d(\frac{p^+}{2\pi\hbar}) d^{D-2}(\frac{\Pi_T}{2\pi\hbar}) \rho^W\big(y^-, \vec{X}_T, p^+, \vec{\Pi}_T)\big)\Big)^{1/2}. \tag{I.5}$$

The natural question that arises is: at what extent the real function $\rho^W(x^-, \vec{x}_T, p^+, \vec{p}_T)$ represents some quantum state, *i.e.* it can be considered to be a Wigner function. The necessary and sufficient conditions are

$$\int dx^- d(\frac{p^+}{2\pi\hbar}) d^{D-2}x_T d^{D-2}(\frac{p_T}{2\pi\hbar}) \rho^W(x^-, \vec{x}_T, p^+, \vec{p}_T)\big[f^* * f\big](x^-, \vec{x}_T, p^+, \vec{p}_T) \geq 0, \tag{I.6}$$

for any $f \in C^\infty(\mathcal{Z}_P^{\mathcal{R}})[[\hbar]]$ and

$$\int dx^- d(\frac{p^+}{2\pi\hbar}) d^{D-2}x_T d^{D-2}(\frac{p_T}{2\pi\hbar}) \rho^W(x^-, \vec{x}_T, p^+, \vec{p}_T) > 0. \tag{I.7}$$

In this appendix we study the situation in which a real function $\rho^W(x^-, \vec{x}_T, p^+, \vec{p}_T)$, satisfying the positivity conditions of Eqs. (I.6) and (I.7), represents the Wigner function of a pure state.

In order to describe this situation in the case of a system of particles we will use the results of Ref. [13]. In our case the solution is quite similar. To begin with we denote

$$\gamma(x^-, \vec{x}_T, y^-, \vec{y}_T) := \int d(\frac{p^+}{2\pi\hbar}) d^{D-2}(\frac{p_T}{2\pi\hbar}) \rho^W(\frac{x^- + y^-}{2}, \frac{\vec{x}_T + \vec{y}_T}{2}, p^+, \vec{p}_T)$$

$$\times \exp\left\{ \frac{i}{\hbar} \big[-(x^- - y^-)p^+ + (\vec{x}_T - \vec{y}_T) \cdot \vec{p}_T \big] \right\}. \tag{I.8}$$

From Eq. (I.2) it follows that if ρ^W is the Wigner function of the pure state $|\Psi\rangle\langle\Psi|$ then the functions γ must satisfy the following equations

$$\frac{\partial^2 \ln \gamma(x^-, \vec{x}_T, y^-, \vec{y}_T)}{\partial x^- \partial y^-} = \frac{\partial^2 \ln \gamma(x^-, \vec{x}_T, y^-, \vec{y}_T)}{\partial x^- \partial y^j}$$

$$= \frac{\partial^2 \ln \gamma(x^-, \vec{x}_T, y^-, \vec{y}_T)}{\partial x^j \partial y^-} = \frac{\partial^2 \ln \gamma(x^-, \vec{x}_T, y^-, \vec{y}_T)}{\partial x^j \partial y^k} = 0 \tag{I.9}$$

for every $j, k = 1, \ldots, D-2$.

Conversely, let γ satisfies Eq. (I.9). The general solution of (I.9) can be factorized as follows

$$\gamma(x^-, \vec{x}_T, y^-, \vec{y}_T) = \Psi_1(x^-, \vec{x}_T)\Psi_2(y^-, \vec{y}_T). \tag{I.10}$$

As the function ρ_W is assumed to be real we get from Eq. (I.8)

$$\gamma^*(x^-, \vec{x}_T, y^-, \vec{y}_T) = \gamma(y^-, \vec{y}_T, x^-, \vec{x}_T). \tag{I.11}$$

Consequently, Eq. (I.10) has the form

$$\gamma(x^-, \vec{x}_T, y^-, \vec{y}_T) = A\Psi_1(x^-, \vec{x}_T)\Psi_1^*(y^-, \vec{y}_T), \tag{I.12}$$

where, by the assumption (I.7), A is a positive real constant. Finally, defining $\Psi := \sqrt{A}\Psi_1(x^-, \vec{x}_T)$ one obtains

$$\gamma(x^-, \vec{x}_T, y^-, \vec{y}_T) = \Psi(x^-, \vec{x}_T)\Psi^*(y^-, \vec{y}_T). \tag{I.13}$$

Substituting $x^- \mapsto x^- + \frac{\xi^-}{2}$, $\vec{x}_T \mapsto \vec{x}_T + \frac{\vec{\xi}_T}{2}$, $y^- \mapsto x^- - \frac{\xi^-}{2}$, $\vec{x}_T \mapsto \vec{x}_T - \frac{\vec{\xi}_T}{2}$, multiplying both sides by $\exp\big\{-\frac{i}{\hbar}(-\xi^- p^+ + \vec{\xi}_T \cdot \vec{p}_T)\big\}$ and integrating with respect to $d(\frac{\xi^-}{2\pi\hbar})d(\frac{\xi}{2\pi\hbar})$ we get exactly the relation (4.23). This means that our function ρ_W is the Wigner function of the pure state $\Psi(x^-, \vec{x}_T)$.

In terms of variables $(x^-, \vec{X}_T, p^+, \vec{\Pi}_T)$ the conditions (I.9) read

$$\frac{\partial^2 \ln\gamma(x^-, \vec{X}_T, y^-, \vec{Y}_T)}{\partial x^- \partial \widetilde{x}^-} = \frac{\partial^2 \ln\gamma(x^-, \vec{X}_T, y^-, \vec{Y}_T)}{\partial x^- \partial \vec{Y}_T}$$

$$= \frac{\partial^2 \ln\gamma(x^-, \vec{X}_T, y^-, \vec{Y}_T)}{\partial \widetilde{x}^- \partial \vec{X}_T} = \frac{\partial^2 \ln\gamma(x^-, \vec{X}_T, y^-, \vec{Y}_T)}{\partial \vec{X}_T \partial \vec{Y}_T} = 0. \tag{I.14}$$

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